Lyapunov-Schmidt Reduction of Differential-Algebraic Equation

Kamal H. Yasir

Department of Mathematical science, College of Education,
Thiqar University

Abstract
In this paper we investigate the asymptotic stability of an equilibrium solution of the differential algebraic equations (DAEs)

$$\dot{x} = f(x, y, \lambda)$$
$$0 = g(x, y, \lambda),$$

effected by the Lyapunov-Schimdt reduction. The main conclusion of this paper is that, under the hypothesis

$$\operatorname{rank} D_y g(x_0, y_0, \lambda_0) = m - 1,$$

the stability or instability of an equilibrium solution (x_0, y_0, λ_0) of the DAEs is determined by the sign of $D_y G$ where G is the reduced function obtained by the Laipunov-Schimdt reduction.

Keywords: Differential algebraic equations, asymptotic stability

1- Introduction
Consider the DAEs

$$\dot{x} = f(x, y, \lambda)$$
$$0 = g(x, y, \lambda) \tag{1.1}$$

where $(f,g): R^n \times R^m \times R^r \to R^n \times R^m$ are C^1. Define the following related sets:

$$\mathcal{M} = \{(x, y, \lambda) \in \ R^n \times R^m \times R^r : g(x, y, \lambda) = 0\}, \tag{1.2}$$

and the set

$$\mathcal{E} = \mathcal{M} \setminus \mathcal{S}, \tag{1.3}$$

where S is defined by

$$\mathcal{S} = \{(x, y, \lambda) \in \ \mathcal{M} : \operatorname{rank} D_y g(x, y, \lambda) = m - 1\}. \tag{1.4}$$

Let $(x_0, y_0, \lambda_0) \in M$ such that $f((x_0, y_0, \lambda_0) = 0$. If rank $D_y g(x_0, y_0, \lambda_0) = m$ then $(x_0, y_0, \lambda_0) \in E$ and it is just a non-degenerate equilibrium point. The degenerate equilibrium points belong to the singular surface S that is the points which satisfy the rank condition

$$\operatorname{rank} D_y g(x, y, \lambda) = m - 1.$$

Since the constraint equation in the DAEs (1.1) is singular at singular point (x_0, y_0, λ_0), the solution may bifurcate at that point, there may be impasse for which the solution does not exist near that point, or the solution is well defined through the singularity. Our study includes the stability of degenerate equilibrium points $(x_0, y_0, \lambda_0) \in S$ of the DAEs for which the solution near that point exists and well is defined. This case is called the stability of singularity induced bifurcation point (SIB) [5], i.e. the equilibrium point on the singular surface S. Let $(x_0, y_0, \lambda_0) \in M$ be an equilibrium point for $\lambda = 0$, i.e. $f(x_0, y_0, 0) = 0$, and that

$$\text{rank } D_y g(x_0, y_0, 0) = m - 1. \tag{1.5}$$

The assumption (1.5) states that zero is an eigenvalue of $D_y g(x_0, y_0, \lambda_0)$. The linearization of the DAEs (1.1) results in the DAEs given by

$$A\dot{\bar{x}} + B\bar{x} = 0 \tag{1.6}$$

where

$$A = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} D_x f^0 & D_y f^0 \\ D_x g^0 & D_y g^0 \end{pmatrix}.$$

Where the script "0" indicates evaluation at the equilibrium point (0, 0, 0). Then because of the rank condition (1.5) the matrix pencil $\{A, B\}$ has an eigenvalue with zero real part.

It is well known that the linearization (1.6) of the DAEs (1.1) yields no information about asymptotic stability if at least an eigenvalue of the matrix pencil $\{A, B\}$ has zero real part. We claim to reduce the DAEs (1.1) to corresponding DAEs with lower dimension by using Lyapunov-Schmidt reduction. Then we may associate such degenerate equilibrium solution of (1.1) with solution of the reduced DAEs

$$\begin{aligned} \dot{x} &= F(x, y, \lambda) \\ 0 &= G(x, y, \lambda) \end{aligned} \tag{1.7}$$

where $(F,G): \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^r \to \mathbb{R}^n \times \mathbb{R}$ is the reduced DAEs obtained by Lyapunov-Schmidt reduction process. Next is to prove under the hypothesis (1.5) the stability or instability of these equilibrium solutions of (1.1) is determined by the sign of $D_y G$, the Jacobian of the reduced function (1.7) with respect to y.

For the proof of our result (Theorem 5.2), we will follow the proof procedure used in [3] for similarly alternative treatments of the stability of solutions of ODEs. So our result here can be considered as an extension of result [3] to DAEs. We emphasise here the result which we obtained that can be applied only to the DEAs (1.1) in case $x \in \mathbb{R}$, $y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^r$. For the case $x \in \mathbb{R}^n$ Theorem 5.1 can't be applied because for determining the asymptotic stability the theorem depends on the sign of the Jacobian G_y of the reduced function $G(x, y, \lambda)$ and when x $\mathbb{R}^n$ there is no meaning of the sign of the matrix G_y. However the result of this paper can be generalized to the case $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$, $\lambda \mathbb{R}^r$ when it is applied to the index-2 Hesenberg DAEs:

$$\begin{aligned} \dot{x} &= f(x, y, \lambda) \\ 0 &= g(x, \lambda) \end{aligned}$$

where $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^n$ and $g: \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ are $\mathbb{C}^1$ with rank condition, $\text{rank } D_x g = n - 1$.

Also the result can be applied to the index-3 Hesenberg DAEs:

$$\begin{aligned} \dot{x} &= f(x, y, \lambda) \\ \dot{y} &= k(x, y, z, \lambda) \\ 0 &= g(x, \lambda) \end{aligned}$$

where $f: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^r \to \mathbb{R}^n$ and $k: \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^s \times \mathbb{R}^r \to \mathbb{R}^m$ $g: \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ are $\mathbb{C}^1$ with rank condition $\text{rank } D_x g = n - 1$.

This paper is organized as follows: In Section 2 we review the theory of asymptotic stability for DAEs. The Lyapunov-Schmidt reduction procedure for DAEs is introduced in Section 3. The equivalence behavior between the DAEs and its reduced DAEs is given in Section 4. Section 5 isdevoted for the main result Theorem 5.2 and we formulated the proof of this theorem.

2- Asymptotic stability theory of DAEs

In this section the review of asymptotic stability theory will be given. For convenient the parameters λ will be dropped in the differential-algebraic equation.

Consider the DAEs (1.1) and suppose that $(x_0, y_0) \in S$ is an equilibrium solution of the DAEs on the singular surface, i.e. the following conditions are satisfied:

$$
\begin{array}{rcl}
f(x_0, y_0) & = & 0 \\
g(x_0, y_0) & = & 0 \\
\operatorname{rank} g_y(x_0, y_0) & = & m - 1
\end{array}
\tag{2.1}
$$

The definition of Lyapunov stability of the equilibrium solution $(x_0, y_0) \in S$ of the DAEs (1.1) is stated in the following definition:

Definition 2.1:

The equilibrium solution $(x_0, y_0) \in S$ of the DAEs (1.1) is stable iff for any $\varepsilon > 0$ there exists a $\delta > 0$ such that if $\| (x(0), y(0)) - (x_0, y_0) \| < \delta$, $\forall (x_0, y_0) \in E$ then $\|(x(t),y(t))-(x_0, y_0)\| < \varepsilon$, $\forall (x(t),y(t)) \in E$, $\forall t \in \mathrm{R}_+$. The equilibrium solution (x_0, y_0) of the DAEs (1.1) is asymptotically stable if it is stable and $\underset{t \to \infty}{Lim} \|(x(t), y(t))\| = (x_0, y_0)$.

The definition of asymptotic stability in terms of the eigenvalues is called linear stability as given in the following definition:

Definition 2.2:

The equilibrium point $(x_0, y_0) \in E$ is linearly stable if every eigenvalue of the matrix pencil$\{A, B\}$has a negative real part, linearly unstable if at least one eigenvalue has positive real part, where

$$
\mathcal{A} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{B} = \begin{pmatrix} D_x f(x_0, y_0) & D_y f(x_0, y_0) \\ D_x g(x_0, y_0) & D_y g(x_0, y_0) \end{pmatrix}.
\tag{2.2}
$$

It is well known that it is asymptotically stable if it is linearly stable and that is unstable if it is linearly unstable [2]. Now if $(x_0, y_0) \in S$ then every eigenvalue of$\{A, B\}$has a negative real part but at least one real part vanishes, then is neither linearly stable nor linearly unstable.

In this situation there is no simple test for asymptotic stability even in using the test of Lyapunov function. So the theory of asymptotic stability mentioned above is not complete.

3-Lyapunov-Schmidt reduction for the DAEs

Consider the DAE (1.1) and assume that the equilibrium point is $(0,0)$ for $\lambda = 0$ such that the conditions (2.1) are satisfied. Let $D_y g(0,0) = B$ then from conditions (2.1)) we have rank$(B)(0,0,0) = m-1$. Choose complements vector spaces H and N to ker$B$ and range$B$ respectively. Then

$$
\mathbb{R}^m = \ker \mathcal{B} \oplus \mathcal{H},
\tag{3.1}
$$

$$
\mathbb{R}^m = \mathcal{N} \oplus \operatorname{range} \mathcal{B}.
\tag{3.2}
$$

Then we conclude that dim $H = \{m-1\}$ and dim $N = 1$. Define the projections $E: \mathrm{R}^m \to$ range$B$ and the complementary projection I-E: $\mathrm{R}^m \to N$ such that the DAEs (1.1) expanded to an equivalent pairs of equations

$$
\begin{array}{rcl}
\dot{x} & = & f(x, y, \lambda), \\
Eg(x, y, \lambda) & = & 0,
\end{array}
\tag{3.3}
$$

and

$$\dot{x} = f(x, y, \lambda),$$
$$(I - E)g(x, y, \lambda) = 0. \tag{3.4}$$

Because of this splitting any vector $y \in \mathrm{R}^m$ can be decomposed in the form $y = v+w$, where $v \in \ker B$ and $w \in H$. Then the equation (3.3) can be written as

$$\dot{x} = f(x, v + w, \lambda),$$
$$Eg(x, v + w, \lambda) = 0. \tag{3.5}$$

Then in (3.5) the second equation can be considered as a map $\phi : \mathrm{R}^n \times \ker B \times H \times \mathrm{R}^r \to$ range B, where

$$\Phi(x, v, w, \lambda) = Eg(x, v + w, \lambda).$$

Now we have

$$\left(\frac{\partial Eg(x, v + w, \lambda)}{\partial w} \right)_{(0,0,0)} = EB$$

Since E act as the identity map on range$B$ so

$$\left(\frac{\partial Eg(x, v + w, \lambda)}{\partial w} \right)_{(0,0,0)} = B$$

and since

$$B : \mathcal{H} \longrightarrow \text{range } B$$

has a full rank at (0,0,0), it follows from the implicit function theorem that the second equation of (3.3) can be solved uniquely for w near (0,0,0), i.e., $w = W(x, v, \lambda)$, where $W : \mathrm{R}^n \times \ker B \times \mathrm{R}^r \to M$ satisfies

$$Eg(x, v + W(x, v, \lambda), \lambda) \equiv 0, \quad W(0, 0, 0) = 0. \tag{3.6}$$

From the second equation of (3.5) and from DAE (1.1) we get the reduced DAEs:

$$\dot{x} = F(x, v, \lambda),$$
$$0 = G(x, v, \lambda), \tag{3.7}$$

where $(F, G) : \mathrm{R}^n \times \ker l B \times \mathrm{R}^r \to \mathrm{R}^n \times N$ defined by:

$$G(x, v, \lambda) = (I - E)g(x, v + W(x, v, \lambda), \lambda), \quad F(x, v, \lambda) = f(x, v + W(x, v, \lambda), \lambda). \tag{3.8}$$

Remark 1.

 The reduced DAEs equation (3.7) has all the information we need from the Lyapunov-Schmidt. The only disadvantage that it maps the second component y between one-dimensional subspaces of R^m.

Remark 2.

 The zeros of (3.7) are in one to one corresponding with the zeros of (1.1) and equation $G(x, v, \lambda) = 0$ is the bifurcation equation.

To see this choose explicit coordinate on ker B and N. For this purpose assume v and v_0^* be none-zero vectors in ker B and (range B)$^\perp$ respectively. Then the vector $v \in \ker B$ can be uniquely written in the form $v = yv_0$ where $y \in \mathrm{R}$. Define $\widetilde{G} : \mathrm{R} \times \mathrm{R}^n \times \mathrm{R}^r \to \mathrm{R}$ by:

$$\widetilde{G}(x, y, \lambda) = \langle v_0^*, G(x, yv_0, \lambda) \rangle,$$

where G is the reduced equation (3.7). It is easy to show that $\widetilde{G}(x, v, \lambda) = 0$ iff $G(x, yv_0, \lambda) = 0$. So the zeros of $\widetilde{G}$ are in one-to-one correspondence with the solutions of $g(x, y, \lambda) = 0$. Then the function $\widetilde{G}$ can be written in terms of the original DAEs (1.1) by using (3.8), i.e,

$$\widetilde{G}(x, y, \lambda) = \langle v_0^*, g(x, yv_0 + W(x, yv_0, \lambda), \lambda) \rangle. \tag{3.9}$$

The function $\widetilde{G}$ is the reduced function to the constraint equation g in the DAEs (1.1) in a new change of coordinates. Also the relation between $\widetilde{G}$ and G is that $\widetilde{G}$ is just a

representation of G in new coordinates. Hence the reduced DAEs in new coordinate are given by

$$\dot{x} = \bar{F}(x, y, \lambda),$$
$$0 = \bar{G}(x, y, \lambda),$$
(3.10)

where $\widetilde{F}, \widetilde{G} : \mathrm{R} \times \mathrm{R}^n \times \mathrm{R}^r \to \mathrm{R}^n$ such that $\widetilde{F}$ defined by

$$\bar{F}(x, y, \lambda) = f(x, yv_0 + W(x, yv_0, \lambda), \lambda)$$
(3.11)

and $\widetilde{G}$ as defined in (3.9). As we mentioned above $\widetilde{G}(x, y, \lambda) = 0$ iff $G(x, yv_0, \lambda) = (I - E)g(x, yv_0 + W(x, yv_0, \lambda), \lambda) = 0$. Thus we have

$$G_y(x, yv_0, \lambda) = (I - E)D_y g(x, yv_0 + W(x, yv_0, \lambda), \lambda)(v_0 + W_y).$$

On evaluating at (0,0,0) we have

$$G_y(0, 0, 0) = (I - E)\mathcal{B}(v_0 + W_y(0, 0, 0)).$$

Since (*I-E*) *B* = 0 so $G_y(0,0,0) = 0$. By a similar way we get $\widetilde{G}_y(0,0,0) = 0$. This means that the reduced DAEs have a singularity at (0,0,0).

4- Equivalence of the stability behavior of the DAEs and the reduced DAEs

In This section we shall show the equivalence between the stability problem for the DAEs (1.1) and the reduced DAEs (3.7). This will be performed by showing that the linearization of the DAEs (1.1) has the same eigenvalues of the linearization of the reduced DAEs (3.7).

Consider the reduced DAEs (3.7) and define the following corresponding set:

$$\bar{\mathcal{M}} = \{(x, y, \lambda) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^r : G(x, y, \lambda) = 0\},$$
(4.1)

and the set

$$\bar{\mathcal{E}} = \bar{\mathcal{M}} \setminus \bar{\mathcal{S}},$$
(4.2)

where

$$\bar{\mathcal{S}} = \{(x, y, \lambda) \in \bar{\mathcal{M}} : G_y(x, y, \lambda) = 0\}.$$
(4.3)

Remark 3.

The relation between those sets and the sets defined in (1.1), (1.2}) *and* (1.3) *is*
$$\bar{\mathcal{M}} \subseteq \mathcal{M}; \bar{\mathcal{E}} \subseteq \mathcal{E}; \bar{\mathcal{S}} \subseteq \mathcal{S}.$$

Remark 4.

The manifold $\widetilde{M}$ is (n+1)-dimensional manifold and will be considered as the reduced manifold of M, whereas for $\widetilde{S}$ will be considered as the singular surface for the reduced DAEs (3.7). In other words, the (n+m)-singular surface S is reduced to the (n+1)-singular surface $\widetilde{S}$ by Lyapunov-Schmidt reduction process. Since $G_y(0,0,0) = 0$ so the singular *point* (0,0,0) *belongs to* $\widetilde{S}$.

Let (0,0,0) be an equilibrium point of the DAEs (1.1) then as we have shown in Section 3 it is also an equilibrium point of the reduced DAEs (3.7). Our purpose is to study the stability of the degenerate equilibrium point (0,0,0) *S* of the DEAs (1.1}) which is associated with the solution of the reduced DAEs (3.7}). To study the stability problem of the solution we need to analyze the linearization of the system about the equilibrium point. The linearization of the reduced DAEs (3.7) about the equilibrium point is

$$\bar{\mathcal{A}}\dot{\bar{x}} + \bar{\mathcal{B}}\bar{x} = 0,$$
(4.4)

where

$$\bar{\mathcal{A}} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \bar{\mathcal{B}} = \begin{pmatrix} D_x F^0 & D_y F^0 \\ D_x G^0 & D_y G^0 \end{pmatrix},$$

where the script "0" indicates evaluation at the equilibrium point. The linearization of the DAEs (1.1) about (0, 0, 0) is:

$$\mathcal{A}\dot{\bar{x}} + \mathcal{B}\bar{x} = 0, \tag{4.5}$$

where

$$\mathcal{A} = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \mathcal{B} = \begin{pmatrix} D_x f^0 & D_y f^0 \\ D_x g^0 & D_y g^0 \end{pmatrix}.$$

Lemma 4.1. Consider the DAEs (1.1}) and its corresponding reduced DAEs (3.7). Assume the rank condition rank$D_y g^0 = m$-1 is satisfied. Then the pencils { $\widetilde{A}$, $\widetilde{B}$ } is singular.

Proof

Since the matrices $\widetilde{A}$ is already singular so it suffices to prove the non-singularly of the matrix $\widetilde{B}$ at (0, 0, 0). For the matrix $\widetilde{B}$ consider

$$\begin{pmatrix} D_x F^0 & D_y F^0 \\ D_x G^0 & D_y G^0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = 0.$$

Solving for *x* and *y* we ge

$$\begin{aligned} x &= (D_x G^0)^{-1} D_y G^0 y, \\ y &= (D_y F^0)^{-1} D_x F^0 x \end{aligned}$$

Since rank$D_y g^0 = m$-1 by the assumption and recall that (*I-E*) *B* = 0, $W_y^0 = 0$ we have

$$D_y G^0 = (I - E)(\mathcal{B}(v_0 + W_y^0)) = 0.$$

Then we have $x = y = 0$ which implies the non-singularly of the matrix $\widetilde{B}$.

From Lemma 4.1 we conclude that the pencil $\{\widetilde{A}, \widetilde{B}\}$ has also zero eigenvalue as the pencil $\{A, B\}$ has. The difference is that the eigenvalues of the pencil $\{A, B\}$\$ satisfies the characteristic polynomial $\{\det(\lambda A + B)\} = 0$ such that degree$\{\det(\lambda A + B)\} = n+m$, whereas the eigenvalues of the pencil $\{\widetilde{A}, \widetilde{B}\}$ satisfies the characteristic polynomial $\{\det(\lambda\widetilde{A} + \widetilde{B}\} = 0$ such that degree$\{\det(\lambda\widetilde{A} + \widetilde{B}\} = n + 1$. Hence the pencil $\{A, B\}$ has $(n+m)$ eigenvalues such that the zero eigenvalue is due to the rank condition rank$D_y g^0 = $ m-1. Consequently The pencil $\{\widetilde{A}, \widetilde{B}\}$ has $(n+1)$ eigenvalues such that *n* none zero eigenvalue and one zero eigenvalue due to $D_y G^0 = 0$. For the index of the reduced DAEs (3.7) in the following lemma we shall see that the Lyapunov-Schimdt reduction does not reduce the index of the DAEs (1.1), i.e. the index of the DAEs is invariant under the Lyapunov-Schimdt reduction process.

Lemma 4.2.

If the DAEs (1.1) is of index 1 then the reduced DAEs (3.7) obtained by Lyapunov-Schmidt reduction is of index 1 also.

Proof

Assume the DAEs (1.1) is of index 1. Differentiate the constraint equation $g(x, y, \lambda) = 0$ with respect to *t* we get

$$D_x g(x, y, \lambda) f(x, y, \lambda) + D_y g(x, y, \lambda)\dot{y} = 0. \tag{4.6}$$

According to the definition of the index concept [6] the DAEs (1.1) will be of index 1 iff (4.6) solved for $\dot{y}$ to get the corresponding ODEs,

$$\begin{aligned} \dot{x} &= f(x, y, \lambda) \\ \dot{y} &= -(D_y g(x, y, \lambda))^{-1} D_x g(x, y, \lambda) f(x, y, \lambda) \end{aligned} \tag{4.7}$$

which defined on the manifold *M*. The ODEs (4.7) is well defined on *M* iff $\{D_y g(x, y, \lambda)\}^{-1}$ exist at every point $(x, y, \lambda) \in M$. Now for the reduced DAEs (3.7) will be of index 1 iff the corresponding ODEs,

$$\begin{aligned} \dot{x} &= F(x, y, \lambda) \\ \dot{y} &= -(D_y G(x, y, \lambda))^{-1} D_x G(x, y, \lambda) F(x, y, \lambda) \end{aligned} \quad (4.8)$$

of index 1 where *F* and *G* as defined in (3.8). We have shown in Section 3 $G(x, y, \lambda)$ is singular if $g(x, y, \lambda)$ is singular. Then we conclude that the reduced DAEs (3.7) is of index 1 also.

5- Asymptotic Stability of the Degenerate Equilibrium Point Via Reduced DAEs

In this section we state the main result Theorem 5.2, which determines the asymptotic stability of the DAEs (1.1). Let $(x_0, y_0, 0)$ be an equilibrium point of the DAEs (1.1) and that *B* = *m*-1 where $B = D_y g(x_0, y_0, 0)$. Since *B* is singular so the equilibrium solution of (1.1) at $(x, y, \lambda) = (x_0, y_0, 0)$ will bifurcate into several equilibrium solutions when $\lambda \neq 0$. In Section 3 we used Lyapunov-Schmidt reduction to associate such bifurcated equilibrium solutions of (1.1) with solutions of a scalar DAEs (3.10) (the scalar here is *y* component). In this section the main result Theorem 5.2 states that under the rank condition, rank*B* = *m*-1 the stability or instability of such bifurcated equilibrium solutions of (1.1) is determined by the sign of $\tilde{G}_y$ the Jacobian of $\tilde{G}$ with respect to *y* where $\tilde{G}$ is the reduced constraint equation (3.9).

Assume the eigenvalues μ_i, i = 1,2,...,*m* of the matrix *B* satisfy

$$\mu_1 = 0, \operatorname{Re}\mu_i < 0, \text{ for } i = 2, ..., m. \quad (5.1)$$

(5.1) means rank*B* = *m*-1. The equilibrium solution (x_0, y_0, λ) of (1.1) will be asymptotically stable if all the eigenvalues of *B* have negative real part; and unstable if at least one eigenvalue has positive real part. Now for (x, y, λ) near (0,0,0) the eigenvalues of $D_y g(x, y, \lambda)$ will be close to those of $D_y g(0,0,0)$. However the eigenvalues of $D_y g(0,0,0)$ satisfy (5.1), so the last *n*-1 eigenvalues will be bounded away from the imaginary axis on an appropriately small neighborhood of (0,0,0) and could not cause (x, y, λ) to be an unstable equilibrium point of (1.1). In contrast, the first eigenvalue (which we denoted by $\mu(x, y, \lambda)$ will be close to zero and might cause such instability. In fact an equilibrium solution (x, y, λ) is linearly stable or unstable according to $\mu(x, y, \lambda)$ is negative or positive respectively. We claim to prove that the reduced constraint function $\tilde{G}(x, y, \lambda)$ obtained from the Lyapunov-Schmidt reduction process has the same sign as $\mu(x, y, \lambda)$.

Before giving the main result the following lemma will be useful in the proof.

Lemma 4.1.

Let $\phi : \psi$,$\mathbb{R}^n \to \mathbb{R}$ be C^∞-maps defined on a neighborhood of zero which is vanish at zero. Assume that

$$\begin{aligned} a. \psi(y) &= 0 \longrightarrow \phi(y) = 0, \\ b. \nabla\phi(0) &\neq 0, \quad \nabla\psi(0) \neq 0, \end{aligned} \quad (5.2)$$

where ∇ indicates gradient. Then $\varphi(y) = \phi(y)/\psi(y)$ is C^∞-map and non-vanishing on some neighborhood of the origin. Moreover $\operatorname{sgn} \varphi(0) = \operatorname{sgn}\langle\nabla\phi(0), \nabla\psi(0)\rangle$.

proof , see[3]

Now The following theorem is the main result which determines the asymptotic stability of the eguilibrium solution of (1.1).

Theorem 5.2.

Let $A\dot{\overline{x}} + B\overline{x} = 0$ be the linearization of DAEs (1.1) and assume that eigenvalues of B satisfy (5.1). Let (3.10) be the reduced DAEs obtained by Lyapunov-Schmidt reduction process. Then the equilibrium solution of the DAEs (1.1) corresponding to solution (x, y, λ) of reduced DAEs (3.10) is asymptotically stable if $\widetilde{G}_y(x, y, \lambda) < 0$ and unstable if $\widetilde{G}_y(x, y, \lambda) > 0$ where $\widetilde{G}(x, y, \lambda) = 0$ is the reduced constraint equation defined in (3.9).

Proof.

The proof of Theorem 5.2 according to Lemma 4.1 requires showing that the quotient

$$\mu(x, y, \lambda)/\tilde{G}_y(x, y, \lambda) \tag{5.3}$$

defines a smooth map which is positive near the origin. Here $\mu(x, y, \lambda)$ is the first eigenvalue of $D_y g(x, y, \lambda)$.

First in order to prove that $\mu(x, y, \lambda)$ is a smooth function of y and λ, we assume that entries of $D_y g(x, y, \lambda)$ vary smoothly with y and λ and that the eigenvalues of a matrix vary smoothly with its entries. So the first eigenvalue $\mu(x, y, \lambda)$ is a smooth function of y and λ thus the first condition of Lemma 4.1 is satisfied. Recall that the equilibrium point of (1.1) corresponding to a solution of 3.10) is asymptotically stable or unstable according as $\mu(x, \Omega, \lambda)$ is positive or negative where $\Omega(x, y, \lambda) = yv_0 + W(x, yv_0, \lambda)$, $v_0 \in$ ker B. For verifying the second condition of the lemma, i.e, to prove

$$\tilde{G}_y(x, y, \lambda) = 0 \longrightarrow \mu(x, \Omega, \lambda) = 0 \tag{5.4}$$

Assume $\widetilde{G}_y(x, y, \lambda) = 0$, for some $(x, y, \lambda) \in R \times R^n \times R^r$. Rewrite the equations (3.6) and (3.9) as

$$Eg(x, \Omega, \lambda) = 0, \tag{5.5}$$

$$\tilde{G}(x, y, \lambda) = \langle v_0^*, g(x, \Omega, \lambda) \rangle. \tag{5.6}$$

Differentiating (5.5) and (5.6) we get

$$E.D_y g(x, \Omega, \lambda).\Omega_y = 0, \tag{5.7}$$

$$\langle v_0^*, D_y g(x, \Omega, \lambda).\Omega_y \rangle = 0. \tag{5.8}$$

Recall from Lyapunov-Schmidt reduction process that: If $u \in R^n \times R^m$ then

$$u = 0 \text{ iff } \langle v_0^*, u \rangle = 0 \text{ and } Eu = 0. \tag{5.9}$$

Then applying (5.9) to (5.7) and (5.8) we get $D_y g(x, \Omega, \lambda).\Omega_y = 0$. This means that zero is an eigenvalue of $D_y g(x, \Omega, \lambda)$ corresponding to the eigenvector Ω_y and since all the other eigenvalues of $D_y g(x, \Omega, \lambda)$ are bounded away from zero we conclude that $\mu(x, \Omega, \lambda) = 0$. For the second condition of Lemma 5.1 we need to prove that $\nabla D_y \widetilde{G}(x, \Omega, \lambda) \neq 0$ and $\nabla \mu(x, \Omega, \lambda) \neq 0$. For these purposes we will use indirect proof, i.e an extra parameter β will be added to the DAEs (1.1) then we get:

$$\begin{aligned} \dot{x} &= \tilde{f}(x, y, \lambda, \beta) \\ 0 &= \tilde{g}(x, y, \lambda, \beta) \end{aligned} \tag{5.10}$$

where $\widetilde{f}(x, y, \lambda, \beta) = f(x, y, \lambda., \beta), \widetilde{g}(x, y, \lambda, \beta) = g(x, y, \lambda., \beta) + \beta y$. The DAEs (5.10) will be considered as the unfolding DAEs to the DAEs (1.1) because for $\beta = 0$ the DAEs (5.10) is equal to the DAEs (1.1). Define $\widetilde{\mu}(x, y, \lambda, \beta)\backslash mu\}$ to be the eigenvalue of $D_y \widetilde{g}(x, y, \lambda, \beta)$ which is close to zero. Then applying the Lyapunov-Schmidt rewduction to the unfolding DAEs (5.10) we get a reduced DAEs

$$\begin{aligned} \dot{x} &= \check{F}(x, y, \lambda, \beta) \\ 0 &= \check{G}(x, y, \lambda, \beta) \end{aligned} \qquad (5.11)$$

where $\check{F}(x, y, \lambda, \beta) = \tilde{f}(x, \tilde{\Omega}(x, y, \lambda, \beta), \lambda, \beta), \check{G}(x, y, \lambda, \beta) = \langle v^*, \tilde{g}(x, \tilde{\Omega}(x, y, \lambda, \beta), \lambda, \beta) \rangle$

and $\tilde{\Omega}(x, y, \lambda, \beta) = yv_0 + W(x, yv_0, \lambda, \beta)$. It is easy to show that the reduced DAEs (5.11) is unfolding to the reduced DAEs (3.10). By a similar way above we can show that

$$D_y \check{G}(x, y, \lambda, \beta) = 0 \longrightarrow \bar{\mu}(x, \tilde{\Omega}, \lambda, \beta) = 0.$$

Now to show that the $\nabla D_y \check{G}(0,0,0) \neq 0$ and $\mu(0,0,0) \neq 0$ we have

$$D_y(\check{G}_\beta)(0,0,0,0) = \langle v_{0*}, D_y(\bar{g}_\beta).v_0 - D^2\bar{g}.v_0.B^{-1}ED_\beta\bar{g} \rangle.$$

Since $D_\beta \tilde{g}(0,0,0,0) = 0$ and $D_y(\tilde{g}_\beta) \cdot v_0 = v_0$ so we have

$$D_y(\check{G}_\beta)(0,0,0,0) = \langle v_0^*, v_0 \rangle.$$

Recall that $v_0 \in \ker B$ and nonzero vector $v_0^* \in (\text{range } B)^\perp$. Hence $\langle v_0^*, v_0 \rangle \neq 0$ and we can choose the vectors v_0 and v_0^* such that $\langle v_0^*, v_0 \rangle > 0$. Then this prove that

$$D_y(\check{G}_\beta)(0,0,0,0) > 0.$$

Next to check the $\nabla \tilde{\mu}(0,0,0,0)$ we have from Lyapunov-Schmidt reduction process that

$$E\bar{g}(x, \tilde{\Omega}(x, y, \lambda, \beta), \lambda, \beta) = 0, \qquad (5.12)$$

where $\tilde{\Omega}(x, y, \lambda, \beta)$ is obtained by solving for $\tilde{\Omega}$ by implicit function theorem. Hence $\tilde{\Omega}$ is unique and since $\tilde{g}(0,0,0,\beta) = 0$ so we have

$$\tilde{\Omega}(0,0,0,\beta) = 0. \qquad (5.13)$$

From (5.10) we have

$$D_y\bar{g}(0,0,0,\beta).v_0 = Bv_0 + \beta v_0 = \beta v_0. \qquad (5.14)$$

Thus β is an eigenvalue of $D_y \tilde{g}(0,0,0,\beta)$ with eigenvector v_0. Since the other eigenvalues of $D_y \tilde{g}(0,0,0,\beta)$ are bounded away from zero so we have

$$\bar{\mu}(0,0,0,\beta) = \beta. \qquad (5.15)$$

Then from (5.13) and (5.15) we conclude that

$$\bar{\mu}(0,\tilde{\Omega},0,\beta) = \beta. \qquad (5.16)$$

By differentiating (5.16) we have

$$\frac{\partial}{\partial \beta}\bar{\mu}(0,\tilde{\Omega},0,0) = 1.$$

Hence the second condition of Lemma 5.1 is satisfied. Then we have

$$\frac{\bar{\mu}(x, \tilde{\Omega}, \lambda, \beta)}{D_y\check{G}(x, y, \lambda, \beta)} \qquad (5.17)$$

is a C^∞ and non-vanishing on some neighborhood of the origin. Moreover we have

$$\frac{\frac{\partial}{\partial\beta}\bar{\mu}(0,\tilde{\Omega},0,0)}{D_y\check{G}_\beta(0,0,0,0)} > 0. \qquad (5.18)$$

Then this prove that the quotient $\tilde{\mu}(x, \tilde{\Omega}, \lambda, \beta)$ and $D_y\check{G}(x, y, \lambda, \beta)$ has the same sign. The proof of Theorem 5.2 will follow on setting $\beta = 0$ in (5.17).

Reference:

1- N.S. Chow and J. Hale, *Method of BifurcationTheory*, Grundlehren, 251, Springer-Verlag, New York(1982).

2- M. Hirsh and S. Smale, *Differential Equation,Dynamical System and Linear Algbra,* Academic Press, New York (1974)

3- M. Golubitsky and D. Shaeffer, *Singulaities and group in Bfurcation Theory*. Vol. 1 Appl. Math. Sci 51 Springer-Verlag New York(1985).

4- H. Kielhofer, *On the Lyapunov stablitiy of stationary solution of semilinear parabolic differentional equations*, J. Diff. Eqns. 922, 193-208(1976).

5- V. Venkatasubramaian, H. Schattle and J. Zaborsky , *Local Bifurcations and Feasibility Regions in Differentail-Algebraic system*, IEEE Trans. On Autom Contr, 40 (12) 1992-2013 (1995).

6- K.E. Brennan, S.L. Campbell and L. R. Petzold, *Numerical Solution of Initial-Value Problem in differential algebraic equations*, North-Holland N.Y. (1989).

الملخص

في هذا البحث درسنا الاستقرارية المطلقة لنقطة الاتزان للمعادلة التفاضلية الجبرية الوسيطية من النوع

$$\dot{x} \equiv f(x, y, \lambda)$$
$$0 \equiv g(x, y, \lambda) \quad , x \in R^n , y \in R^m$$

بواسطة استخدام طريقة لبايونوف للتقليص (Lyapunov Schmdit Reduction) .

النتيجة الرئيسية للبحث هي انه استقرارية وعدم استقرارية نقطة الاتزان (x_0, y_0, λ_0) للمعادلة أعلاه يمكن معرفتها بمعرفة إشارة المصفوفة $D_y G$ حيث G هي الدالة المقلصة والتي يمكن الحصول عليها من طريقة ليابونوف المستخدمة في هذا البحث وتحت شرط :

$$rank \, D_y g(x_0, y_0, \lambda_0) \equiv m - 1$$