Asymptotic Behavior of Derivatives of Near Best Approximation Polynomials

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Abstract

We Will relate the Behavior of derivatives of the near best approximation to the Ditzian - Totik modulus of smoothness of the functions in $L_p, 0 < p < 1$.

Key Words: Unconstrained approximation field , measurable function , maximal function .

1-Introduction and the Auxiliary Results

In the unconstrained approximation field we have a question: Is there any relation between the nearly best polynomial n th degree approximation to functions in the $L_p[-1,1]$ for $p < 1$ to the Ditzian Totik modulus of smoothness? The prime interest of this paper is to provide answer to the above question. Let Π_n denote the set of all algebraic polynomials of degree $\leq n$. $L_p(I)$, $I = [-1,1]$ consisting of the collection of all equivalence classes under equality a.e. for which

$$\|f\|_p := \left(\int_{-1}^{1} |f(x)|^p dx \right)^{1/p} < \infty.$$

Since we need the L_p quasi norm on other intervals we will in all cases of an interval $J \neq [-1,1]$, indicate that by writing $\|f\|_{L_p(J)}$.

If $\varphi(x) = \sqrt{1 - x^2}$ *the mth order Ditzian Totik Modulus* $\omega_\varphi^m(f, \delta)$ is given by (see [5],or [1] for example)

$$\omega_\varphi^m(f, \delta, I)_p := \sup_{0 < h \leq \delta} \left\| \Delta_{h\varphi(x)}^m (f, x) \right\|_{L_p(I)}, \delta > 0$$

where

$$\Delta_h^m(f,x,I) := \Delta_h^m(f,x) := \begin{cases} \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} f\left(x - \frac{mh}{2} + ih\right), & x \pm \frac{mh}{2} \in I \\ 0, & o.w. \end{cases},$$

is the *symmetric mth difference*.

For $f \in L_p(I)$, the *rate of unconstrained approximation* is defined by

$$E_n(f)_p := \inf_{p_n \in \Pi_n} \|f - p_n\|_p .$$

If f is a function in $L_p(I)$, $0 < p < 1$ then a polynomial p_n of degree n is a *near best L_p approximation* (with constant M) to f from among all polynomials of degree $\le k$ if

$$\|f - p_n\|_p \le M E_n(f)_p ,$$

if $M = 1$ then p_n is a best approximant.

We shall make use of the following lemmas that were proved in [2], [4], [1] respectively

<u>Lemma 1.1</u> *For $f \in L_p(I), 0 < p < 1$ and k a positive integer, there exists an algebraic polynomial p_n of degree $\le n$ such that*

$$\|f - p_n\|_p \le c(p)\omega_\varphi^k(f, n^{-1})_p .$$

<u>Lemma 1.2</u> *For a function $f \in L_p(I)$, $0 < p < \infty$, and $r \in N$ we have*

$$\omega_\varphi^r(f,\delta,[a,b])_p \approx \widetilde{K}_{r,\varphi}(f,\delta^r)_p, \delta > 0$$

where $\widetilde{K}_{r,\varphi}$ is the Ditzian-Totik $\widetilde{K}$ −functional [4], defined by

$$\widetilde{K}_{r,\phi}(f,\delta^r)_p := \inf_{\substack{p_n \in \Pi_n \\ n = \left[\frac{1}{\delta}\right]}} \left\{ \|f - p_n\|_{L_p(J)} + \delta^r \left\|\phi^r p_n^{(r)}\right\|_{L_p(J)} \right\},$$

where [k] is the least integer greater than k.

<u>Lemma 1.3</u>. *Suppose for $m \in N$, that $\left\|\varphi^m p_n^{(m)}\right\|_p \le cn^m \sigma\left(\frac{1}{n}\right)$, where p_n is the best nth degree polynomial approximation to f in L_p, $p < 1$, and $\sigma(u) \to 0$ as $u \to 0+$. Then*

$$E_n(f)_p \le c(p) \int_o^{1/n} \frac{\sigma(u)}{u} du ,$$

and

$$\omega_\varphi^m(f, n^{-1})_p \le c(p) \int_o^{1/n} \frac{\sigma(u)}{u} du .$$

In particular if

$$\int_{o}^{1/n} \frac{\sigma(u)}{u} du = O\left(\sigma\left(\frac{1}{n}\right)\right),$$

the above implies

$$E_n(f)_p \le c(p)\sigma\left(\frac{1}{n}\right),$$

and

$$\omega_\varphi^m\left(f, n^{-1}\right)_p \le c(p)\sigma\left(\frac{1}{n}\right).$$

We mean by $f(x) = \mathrm{O}(g(x))$ as $x \to A$, *A* is constant , that $\frac{|f(x)|}{|g(x)|} \le k$ with *k* constant as $x \to A$.

Throughout this paper *c*s denote constants not depending on *n* and $f(x)$, the value of *c*s may vary at different places, even on the same line. In order to emphasize *c* depends only on parameters *v and w* the notation $c(v,w)$ is used.

1. The Main Results

Our main results in this article are

Theorem 2.1. ***For any 0<p<1 and any polynomial*** $p_n \in \Pi_n$ ***satisfying*** $\|f - p_n\|_p \le ME_n(f)_p$ ***with fixed*** M ***,and integer*** $r > 0$ ***we have***

$$\left\|\phi^r p_n^{(r)}\right\|_p \le c(p)\omega_\phi^r\left(f, n^{-1}\right)_p. \qquad \textbf{(2.2)}$$

Proof. First we prove the estimate

$$\left\|\varphi^{r+1} p_n^{(r+1)}\right\|_p \le cn^{r+1}\omega_\varphi^r\left(f, n^{-1}\right)_p. \qquad (2.3)$$

For ℓ given by $\ell = \max\{k : 2^k < n\}$ we expand $p_n(x)$ by

$$p_n(x) - p_0(x) = p_n(x) - p_{2^\ell}(x) + \left(p_{2^\ell}(x) - p_{2^{\ell-1}}(x)\right) + \cdots + \left(p_1(x) - p_0(x)\right)$$

We recall that for $m < n, \|p_n - p_m\|_p^p \le 2ME_m(f)_p^p$ and use $\left\|\left(1 - x^2\right)^{r/2} p_n^{(r)}(x)\right\|_p \le c(p)n^r \|p_n(x)\|_p, p < 1$ with $r + 1$ instead of r to obtain:

$$\left\|\varphi^{r+1} p_n^{(r+1)}\right\|_p^p \le c(p)n^{r+1}\sum_{k=0}^{\ell} 2^{k(r+1)p} E_{2^k}(f)_p^p.$$

The inequalities in Lemma 1.1 and 1.2 lead to

$$\begin{aligned} E_{2^k}(f)_p^p &\le c(p)\omega_\phi^r\left(f, 2^{-k}\right)_p^p \\ &\le c(p)\widetilde{K}_{r,\varphi}\left(f, 2^{-kr}\right)_p^p \end{aligned}$$

$$\le c(p)2^{pr+p(\ell-k)r}\widetilde{K}_{r,\varphi}\left(f,2^{-(\ell+1)r}\right)_p^p$$
$$\le c(p)2^{p(\ell-k)r}\,\omega_\phi^r\left(f,2^{-\ell}\right)_p^p.$$

Combining the above we write

$$\left\|\phi^{r+1}p_n^{(r+1)}\right\|_p^p \le c(p)\sum_{k=0}^{\ell}2^{k(r+1)p+(\ell-k)rp}\,\omega_\phi^r\left(f,2^{-\ell}\right)_p^p$$
$$\le c(p)n^{(r+1)p}\omega_\varphi^r\left(f,n^{-1}\right)_p^p.$$

For $\varphi(x)=\sqrt{1-x^2}$ it follows from a well-known result on difference that

$$n^{-r}\left(1-x^2\right)^{\frac{r}{2}}p_n^{(r)}(\xi(x))=\Delta^r_{\frac{\varphi(x)}{n}}p_n(x)$$

for some $\xi(x)$ satisfying $x-\frac{r\varphi(x)}{2n}\le\xi(x)\le x+\frac{r\varphi(x)}{2n}$. To estimate $\Delta^r_{\frac{\varphi(x)}{n}}p_n(x)$ in $L_p(D_n)$ where $D_n=\left[-1+\frac{2r^2}{n^2},1-\frac{2r^2}{n^2}\right]$ which guarantees $x\pm\frac{r\phi(x)}{2n}\in\left[-1+\frac{r^2}{n^2},1-\frac{r^2}{n^2}\right]$ for $x\in D_n$ we write:

$$\left\|\Delta^r_{\frac{\varphi}{n}}p_n\right\|^p_{L_p(D_n)}\le\left\|\Delta^r_{\frac{\varphi}{n}}(p_n-f)\right\|^p_{L_p(D_n)}+\left\|\Delta^r_{\frac{\varphi}{n}}f\right\|^p_{L_p(D_n)}$$
$$\le\omega_\varphi^r\left((p_n-f),n^{-1},D_n\right)_p^p+\omega_\varphi^r\left(f,n^{-1}\right)_p^p$$
$$\le c(p)\left\|p_n-f\right\|^p_{L_p(D_n)}+\omega_\varphi^r\left(f,n^{-1}\right)_p^p$$
$$\le c(p)E_n(f)_p^p+\omega_\varphi^r\left(f,n^{-1}\right)_p^p$$
$$\le c(p)\omega_\varphi^r\left(f,n^{-1}\right)_p^p.$$

To obtain (2.2) we write (note that $p_n^{(r)}(\xi(x))$ is a measurable function of x)

$$\left\|n^{-r}\varphi^r p_n^{(r)}\right\|^p_{L_p(D_n)}\le\left\|n^{-r}\varphi^r p_n^{(r)}-\Delta^r_{\frac{\varphi}{n}}p_n\right\|^p_{L_p(D_n)}+\left\|\Delta^r_{\frac{\varphi}{n}}p_n\right\|^p_p$$
$$=\left\|n^{-r}\varphi^r p_n^{(r)}-n^{-r}\varphi^{\frac{r}{2}}p_n^{(r)}\right\|^p_{L_p(D_n)}+\left\|\Delta^r_{\frac{\varphi}{n}}p_n\right\|^p_p$$

$$\le n^{-r}\left\|\varphi^{\frac{r}{2}}\left(p_n^{(r)}-p_n^{(r)}(\xi(x))\right)\right\|_{L_p(D_n)}^p+\left\|\Delta_{\frac{\varphi}{n}}^r p_n\right\|_p^p .$$

$$\le\left\|\left(1-x^2\right)^{\frac{r}{2}}\int_x^{\xi(x)}p_n^{(r+1)}(u)du\right\|_{L_p(D_n)}^p+c(p)\omega_\varphi^r\left(f,n^{-1}\right)_p^p$$

$$=: E_1+c(p)\omega_\varphi^r\left(f,n^{-1}\right)_p^p .$$

We have for $L_p(D_n)$:

$$E_1\le\left\|\left(1-x^2\right)^{\frac{r}{2}}\int_{x-\frac{r\ (x)}{2n}}^{x+\frac{r\ (x)}{2n}}\left|p_n^{(r+1)}(u)\right|du\right\|_{L_p(D_n)}^p =: I_p ,$$

for $0<p<1$ we can deduce from (2.3), recalling that the maximal function $M(F)$ satisfies $\|M(F)\|_p\le c(p)\|F\|_p$.

$$I_p\le\frac{r}{n}\left\|\varphi(x)^{r+1}\frac{n}{r\varphi(x)}\int_{x-\frac{r\varphi(x)}{2n}}^{x+\frac{r\varphi(x)}{2n}}\left|p_n^{(r+1)}(u)\right|du\right\|_{L_p(D_n)}^p$$

$$\le\frac{c(p)}{n}\left\|M\left(\varphi^{r+1}p_n^{(r+1)},x\right)\right\|_{L_p(D_n)}^p$$

$$\le c(p)\left\|\varphi^{r+1}p_n^{(r+1)}\right\|_p^p$$

$$\le c(p)\omega_\varphi^r\left(f,n^{-1}\right)_p^p .$$

Hence using the following inequality from [2] we can obtain (2.2).

$$\left\|\varphi^r P_n\right\|_{L_p(D_n)}\le c\left\|\varphi^r P_n\right\|_{L_p[-1,1]}\quad\text{for}\quad P_n\in\Pi_n .$$

This completes the proof of theorem 2.1 ◘

We can state the following consequence to Theorem 2.1 and Lemma 1.3

Corollary 2.4. ***For any p<1 and*** $0<\alpha\le r$ ***,we have***

$$\left\|\varphi^r p_n^{(r)}\right\|_p\le c(p)n^{r-\alpha}\ \textit{and}\ \ \omega_\varphi^r\left(f,n^{-1}\right)_p\le c(p)n^{-\alpha}$$

are equivalent.

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سلوك مشتقات
متعددات حدود التقريب المقارب الأفضل
للدوال في الفضاءات L_p عندما $0<p<1$

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المستخلص

هنالك نظريات عديدة تربط بين رتبة التقريب الأفضل باستخدام متعددات الحدود الجبرية و مقياس نعومة دالة في فضاء ما. خَطَرَ على بالنا التساؤل الأتي: هل من الممكن إيجاد عَلاقات تربط التقريب الأفضل أو التقريب المقارب الأفضل للدالة مع مقياس نعومتها وخاصة للدوال في الفضاءات L_p عندما $(0<p<1)$ ؟. في هذا البحث قمنا بإيجاد نظرية تربط بين مشتقات متعددات حدود التقريب الجانبي الأفضل للدوال في الفضاءات L_p $(0<p<1)$ مع مقياس النعومة ω_φ^r.