High Order Finite Difference Scheme for One Dimensional Heat Transport Equation at the Microscale.

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Abstract

A sixth order compact difference scheme for space with Crank-Nicholson scheme for time is employed to solve one dimensional microscale heat transport equation. The unconditionally stability of this new version of finite difference scheme is proved with respect to initial values. Numerical experiments are introduced to test the accuracy of the sixth order compact finite differences and compare it with both the second order and fourth order difference schemes. Our Numerical results showed that the sixth order compact scheme is computationally more efficient and more accurate than the second and fourth order schemes.

Key words: finite differences, sixth order compact, 1D heat transport equation, Crank-Nicholson method.

1. Introduction

Microtechnologies based on high rate heating on thin film structure have received much attention in recent years due to the advancement of short pulse laser technologies and their applications to micro manufacturing processes [1, 6]. These microtechnology applications frequently deals with thermal behavior of thin films [4].

Many applications can be modeled by the microscale heat transport equation, for example phonon electron interaction model [9], the single energy equation [11, 12], the phonon radiative transfer model [3], and the lagging behavior model [7, 10, 11].

Few authors deal with the numerical solution of one dimension microscale heat transport equation. By using Crank-Nicholson technique Qui and Tien [8] solve the phonon electron interaction model. Joshi and Majumdar [4] use an explicit upstream difference method to solve the phonon radiative transfer model in a one dimensional medium. Zhang and Zhao [14] consider a fourth order compact finite difference discretization scheme to solve the one dimension microscale heat transport equation and prove this new scheme is unconditionally stable with initial values.

Zhang and Zhao in [15] and [16] solve the two and three dimension microscale heat transport equation, respectively, using second order for time and space.

Heat transport equation concerned with thermal behavior of thin films is described by [7, 13]:

$$\frac{\partial^2 \tilde{T}}{\partial \tilde{x}^2} + \tau_T \frac{\partial^3 \tilde{T}}{\partial \tilde{x}^2 \partial \tilde{t}} = \frac{1}{\alpha} \frac{\partial \tilde{T}}{\partial \tilde{t}} + \frac{\tau_q}{\alpha} \frac{\partial^2 \tilde{T}}{\partial \tilde{t}^2} \quad , \tag{1}$$

Where $\tilde{T}$ is the temperature, α, τ_T, and τ_q are positive constants. Note that τ_T and τ_q are the time lags of the heat flux and temperature gradient, respectively.

For simplification, we rewrite equation (1) in a more general and compact form as in [11]:

$$\frac{\partial^2 T}{\partial x^2} + p\frac{\partial^3 T}{\partial x^2 \partial t} = R\frac{\partial T}{\partial t} + S\frac{\partial^2 T}{\partial t^2}, \tag{2}$$

where
$$T(\tilde{x},\tilde{t}) = \frac{\tilde{T}(\tilde{x},\tilde{t}) - \tilde{T}(0,\tilde{t})}{\tilde{T}(\tilde{L},\tilde{t}) - \tilde{T}(0,\tilde{t})} \quad , \quad x = \frac{\tilde{x}}{2\sqrt{\alpha\tau_q}} , \quad t = \frac{\tilde{t}}{2\tau_q} ,$$

$p = \dfrac{\tau_T}{2\tau_q}$, R and S are positive constants.

The boundary and initial conditions are given as

$$T(0,t) = T_0 , \qquad T(L,t) = T_1 , \qquad t > 0,$$

and

$$T(x,0) = T_2 , \qquad \frac{\partial T}{\partial t}(x,0) = T_3 , \qquad 0 \le x \le L.$$

To avoid the three level time discretization, rewrite (2) as

$$\frac{\partial^2}{\partial x^2}\left(T + P\frac{\partial T}{\partial t}\right) = \frac{\partial}{\partial t}\left(RT + S\frac{\partial T}{\partial t}\right) . \tag{3}$$

Define an intermediate function as;

$$\theta = RT + S\frac{\partial T}{\partial t} . \tag{4}$$

1- If $S - PR \ge 0$, Eq.(4) becomes

$$\frac{\partial T}{\partial t} = \frac{1}{S}(\theta - RT) . \tag{5}$$

Substituting (4) and (5) in (3) and after simplification, we obtain

$$\frac{\partial^2}{\partial x^2}\left[(S - PR)T + P\theta\right] = S\frac{\partial\theta}{\partial t} . \tag{6}$$

2- If $S - PR < 0$, from (4) we have

$$T = \frac{1}{R}\left(\theta - S\frac{\partial T}{\partial t}\right) . \tag{7}$$

Substituting (4) and (7) into (3), and after simplification, we obtain

$$\frac{\partial^2}{\partial x^2}\left[\theta + (PR - S)\frac{\partial T}{\partial t}\right] = R\frac{\partial\theta}{\partial t} . \tag{8}$$

Furthermore, the initial and boundary conditions written as:

$$T(x,0) = T_2, \quad \theta(x,0) = RT_2 + ST_3, \qquad 0 \le x \le L,$$
and
$$T(0,t) = T_0, \quad \theta(0,t) = RT_0 + S\frac{\partial T_0}{\partial t}, \quad T(L,t) = T_1, \quad \theta(L,t) = RT_1 + S\frac{\partial T_1}{\partial t}, \quad t > 0.$$

In this paper we consider the microscale heat transport equation in one dimension. We introduce a sixth order accuracy for space and second order for time as development to Zhang and Zhao work in [14]. We prove that this new scheme is unconditionally stable with initial values. The results of sixth order are compared with the results of Zhang and Zhao in [14].

2. Sixth order compact discretization

The domain $[0,1] \times [0,T]$ is divided into $N \times M$ mesh with the spatial step size $\Delta x = 1/N$ in x and the time size $\Delta t = T/M$.

We denote $T(i\Delta x, n\Delta t)$, conveniently, by T_i^n, and the grid points (x_i, t_n) are defined by $x_i = i\Delta x$, $t_n = n\Delta t$.

We denote the finite difference scheme based on the second order central difference scheme by

$$\left.\frac{\partial^2 u_i}{\partial x^2}\right|_i = \delta_x^2 u_i + O(\Delta x^2) = \frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x)^2,$$

$$\left.\frac{\partial^4 u_i}{\partial x^4}\right|_i = \delta_x^2 \delta_x^2 u_i + O(\Delta x)^2 = \delta_x^2\left(\frac{u_{i+1} - 2u_i + u_{i-1}}{\Delta x^2} + O(\Delta x)^2\right),$$

By second order finite difference, we can discretizating Poisson equation

$$u_{xx} = f, \tag{9}$$

As

$$\delta_x^2 u_i = f_i + O(\Delta x)^2.$$

Three points fourth order compact scheme to approximate (9) is given by

$$\delta_x^2 u_i = (1 + \frac{(\Delta x)^2}{12}\delta_x^2)f_i + O(\Delta x)^4. \tag{10}$$

Similarly we discretize equation (9) using sixth order compact scheme:

$$\delta_x^2 u_i = f_i + \frac{(\Delta x)^2}{12}\delta_x^2 f_i + \frac{(\Delta x)^4}{360}\delta_x^4 f_i + O(\Delta x)^6. \tag{11}$$

Hence, we can use the sixth order compact scheme (11) and formula (6) by considering $u = (S - PR)T + P\theta$ and $f = S\frac{\partial \theta}{\partial t}$, assuming $S - PR \ge 0$, we have

$$\frac{1}{2}\delta_x^2\left[(S - PR)T_i^{n+1} + P\theta_i^{n+1}\right] + \frac{1}{2}\delta_x^2\left[(S - PR)T_i^n + P\theta_i^n\right] =$$

$$S\left[\frac{\partial\theta}{\partial t}+\frac{(\Delta x)^2}{12}\delta_x^2\frac{\partial\theta}{\partial t}+\frac{(\Delta x)^4}{360}\delta_x^4\frac{\partial\theta}{\partial t}\right]_i. \tag{12}$$

$$\text{Let } u=\frac{\partial^2\theta}{\partial x^2}, \tag{13}$$

Eq.(12) can be written as

$$\frac{1}{2}\delta_x^2\left[(S-PR)T_i^{n+1}+P\theta_i^{n+1}\right]+\frac{1}{2}\delta_x^2\left[(S-PR)T_i^n+P\theta_i^n\right]=$$
$$S\left[\frac{\partial\theta}{\partial t}+\frac{(\Delta x)^2}{12}\delta_x^2\frac{\partial\theta}{\partial t}+\frac{(\Delta x)^4}{360}\delta_x^2\frac{\partial u}{\partial t}\right]_i, \tag{14}$$

Since $\quad\dfrac{\partial\theta}{\partial t}\bigg|_i=\dfrac{\theta_i^{n+1}-\theta_i^n}{\Delta t}, \quad$ and $\quad\dfrac{\partial u}{\partial t}\bigg|_i=\dfrac{u_i^{n+1}-u_i^n}{\Delta t}.$

(14) can be written as follows

$$\frac{1}{2}\delta_x^2\left[(S-PR)T_i^{n+1}+P\theta_i^{n+1}\right]+\frac{1}{2}\delta_x^2\left[(S-PR)T_i^n+P\theta_i^n\right]=$$
$$S\left[\frac{\theta_i^{n+1}-\theta_i^n}{\Delta t}+\frac{(\Delta x)^2}{12}\delta_x^2\frac{\theta_i^{n+1}-\theta_i^n}{\Delta t}+\frac{(\Delta x)^4}{360}\delta_x^2(\frac{u_i^{n+1}-u_i^n}{\Delta t})\right]. \tag{15}$$

For computing T_i^{n+1}, we discretize (4) using the trapezoidal method as

$$\frac{1}{2}(\theta_i^{n+1}+\theta_i^n)=\frac{R}{2}(T_i^{n+1}+T_i^n)+S(\frac{T_i^{n+1}-T_i^n}{\Delta t}). \tag{16}$$

Rearrange (16) to get

$$(S+\frac{\Delta tR}{2})T_i^{n+1}=\frac{\Delta t}{2}(\theta_i^{n+1}+\theta_i^n)+(S-\frac{R\Delta t}{2})T_i^n.$$

Dividing both sides by $(S+\dfrac{R\Delta t}{2})$, yields

$$T_i^{n+1}=(S-\frac{R\Delta t}{2})(S+\frac{R\Delta t}{2})^{-1}T_i^n+\frac{\Delta t}{2}(S+\frac{R\Delta t}{2})^{-1}(\theta_i^{n+1}+\theta_i^n). \tag{17}$$

This can be used to find a value for T_i^{n+1}. For computing u_i^{n+1}, we discretize (13) using central finite difference method and Crank-Nicholson Technique, we get

$$u_i^{n+1}+u_i^n=\delta_x^2(\theta_i^{n+1}+\theta_i^n). \tag{18}$$

The last equation can be used to find a value for u_i^{n+1}. Substituting (17) into (15), we get

$$S\left[\theta_i^{n+1} - \theta_i^n + \frac{(\Delta x)^2}{12}\delta_x^2(\theta_i^{n+1} - \theta_i^n) + \frac{(\Delta x)^4}{360}\delta_x^2(u_i^{n+1} - u_i^n)\right] =$$

$$\frac{\Delta t}{2}\delta_x^2\left[(S-PR)T_i^n + P\theta_i^n\right] + \frac{\Delta t}{2}\delta_x^2(S-PR)(S+\frac{R\Delta t}{2})^{-1}(S-\frac{R\Delta t}{2})T_i^n$$

$$+ \frac{\Delta t}{2}\delta_x^2(S-PR)(S+\frac{R\Delta t}{2})^{-1}\frac{\Delta t}{2}(\theta_i^{n+1} + \theta_i^n) + \frac{\Delta t}{2}P\delta_x^2\theta_i^{n+1}.$$

Multiplying both sides by $(S+\frac{R\Delta t}{2})$ and rearranging equation yields

$$(S+\frac{R\Delta t}{2})\left[\theta_i^{n+1} - \theta_i^n + \frac{(\Delta x)^2}{12}\delta_x^2(\theta_i^{n+1} - \theta_i^n) + \frac{(\Delta x)^4}{360}\delta_x^2(u_i^{n+1} - u_i^n)\right] =$$

$$(S-PR)\Delta t\delta_x^2 T_i^n + \frac{\Delta t}{2}(P+\frac{\Delta t}{2})\delta_x^2(\theta_i^{n+1} + \theta_i^n). \qquad (19)$$

Eq.(19) can be used to compute θ_i^{n+1}. The truncation error is $O\left((\Delta t)^2 + (\Delta x)^6\right)$, and the method is said to be of sixth order accurate. The same discretization scheme will be obtained if we assume $(S-PR) < 0$, and use (4) and (6) to the derivative. Now, we need to discretize the boundary and initial conditions as

$$T_0^n = (T_0)^n, \quad \theta_0^n = (RT_0 + S\frac{\partial T_0}{\partial t})^n, \quad T_N^n = (T_1)^n, \quad \theta_N^n = (RT_1 + S\frac{\partial T_1}{\partial t})^n, \quad (20)$$

$$T_i(x,0) = (T_2)_i, \quad \theta_i(x,0) = (RT_2 + ST_3)_i, \qquad 0 \le x \le L. \qquad (21)$$

3. Stability

Now, we discuss the stability of the schemes (17), (18) and (19), by using the discrete energy method [2, 5, 14]. The set of $\{u^n = \{u_i^n\}, with \ u_0^n = u_N^n = 0\}$ denoted by Ω. We make the following definitions for any $u^n, v^n \in \Omega$:

$$(u^n, v^n) = \Delta x\sum_{i=1}^{N-1} u_i^n v_i^n, \qquad \left\|u^n\right\|^2 = (u^n, u^n),$$

$$\left\|\nabla_x u^n\right\|^2 = (\nabla_x u^n, \nabla_x u^n) = \Delta x\sum_{i=1}^{N-1}(\nabla_x u_i^n)^2, \qquad (22)$$

and

$$G(u^n) = \left\|u^n\right\|^2 - \frac{\Delta x^2}{12}\left\|\nabla_x u^n\right\|^2, \qquad (23)$$

where

$$\nabla_x u_i^n = \frac{u_{i+1}^n - u_i^n}{\Delta x}.$$

is the forward difference operators.

To prove that our scheme is unconditionally stable will used the following two lemmas:

Lemma 1 [14]: For any $u^n, v^n \in \Omega$, the following equality holds:

$$\left(\delta_x^2 u^n, v^n\right) = -\left(\nabla_x u^n, \nabla_x v^n\right),$$

Lemma 2 [14]: For any $u^n \in \Omega$, the following inequality holds.

$$\frac{2}{3}\|u^n\|^2 \le G(u^n) \le \left(1 - \frac{1}{3N^2}\right)\|u^n\|^2. \qquad (24)$$

In the next theorem, we prove that our scheme is unconditionally stable with respect to initial values.

Theorem: Assume that $\{T_i^n, \theta_i^n, u_i^n\}$ and $\{\tilde{T}_i^n, \tilde{\theta}_i^n, \tilde{u}_i^n\}$ are solutions of the discretization schemes (17), (18) and (19) with initial values $\{T_i^0, \theta_i^0, u_i^0\}$ and $\{\tilde{T}_i^0, \tilde{\theta}_i^0, \tilde{u}_i^0\}$ respectively. Let $V_i^n = \theta_i^n - \tilde{\theta}_i^n$, $\eta_i^n = u_i^n - \tilde{u}_i^n$, $\psi_i^n = T_i^n - \tilde{T}_i^n$ from Eqs.(17) ,(18) and (19), we obtain that V_i^n, η_i^n, ψ_i^n Satisfy

$$\frac{2}{3}\|V^{n+1}\|^2 + \frac{\Delta x^4}{360}\|\eta_i^{n+1}\|^2 + (S - PR)\|\nabla_x \psi^{n+1}\|^2 \le \|V^0\|^2 + \frac{\Delta x^4}{360}\|\eta_i^0\|^2 + (S - PR)\|\nabla_x \psi^0\|^2,$$
$$\text{if } (S - PR) > 0. \quad (25)$$

$$\frac{2}{3}\|V^{n+1}\|^2 + \frac{\Delta x^4}{360}\|\eta_i^{n+1}\|^2 + (PR - S)\|\nabla_x \psi^{n+1}\|^2 \le \|V^0\|^2 + \frac{\Delta x^4}{360}\|\eta_i^0\|^2 + (PR - S)\|\nabla_x \psi^0\|^2,$$
$$\text{if } (S - PR) < 0. \quad (26)$$

for any $0 \le n\Delta t \le t_{stop}$. This implies that the finite difference scheme is unconditionally stable with respect to the initial values.

Proof: Here, we prove (25) for $S - PR \ge 0$. The proof of (26) with $S - PR < 0$ can be done analogously and is therefore omitted here.

We rewrite (15) as

$$S\left[\frac{\theta_i^{n+1} - \theta_i^n}{\Delta t} + \frac{(\Delta x)^2}{12}\delta_x^2 \frac{\theta_i^{n+1} - \theta_i^n}{\Delta t} + \frac{(\Delta x)^4}{360}\delta_x^2 (\frac{u_i^{n+1} - u_i^n}{\Delta t})\right] =$$
$$\frac{1}{2}\delta_x^2 (S - PR)(T_i^{n+1} + T_i^n) + \frac{1}{2}\delta_x^2 P(\theta_i^{n+1} + \theta_i^n).$$

Since $\{T_i^n, \theta_i^n, u_i^n\}$ and $\{\tilde{T}_i^n, \tilde{\theta}_i^n, \tilde{u}_i^n\}$ are solutions of the discretization schemes then $V_i^n, \psi_i^n, \eta_i^n \in \Omega$ satisfy

$$S\left[\frac{V_i^{n+1} - V_i^n}{\Delta t} + \frac{(\Delta x)^2}{12}\delta_x^2 \frac{V_i^{n+1} - V_i^n}{\Delta t} + \frac{(\Delta x)^4}{360}\delta_x^2 (\frac{\eta_i^{n+1} - \eta_i^n}{\Delta t})\right] =$$
$$\frac{1}{2}\delta_x^2 (S - PR)(\psi_i^{n+1} + \psi_i^n) + \frac{1}{2}\delta_x^2 P(V_i^{n+1} + V_i^n). \qquad (27)$$

From (16)

$$\frac{1}{2}(V_i^{n+1} + V_i^n) = \frac{R}{2}(\psi_i^{n+1} + \psi_i^n) + S(\frac{\psi_i^{n+1} - \psi_i^n}{\Delta t}). \qquad (28)$$

and from (18)

$$\eta_i^{n+1} + \eta_i^n = \delta_x^2 (V_i^{n+1} + V_i^n).$$

(29)

Multiplying Eq. (27) by $(V_i^{n+1} + V_i^n)\varDelta x$ and summing i from 1 to $N-1$ and using Lemma 1, yield

$$\frac{S}{\varDelta t}[\|V^{n+1}\|^2 - \|V^n\|^2 - \frac{(\varDelta x)^2}{12}(\|\nabla_x V^{n+1}\|^2 - \|\nabla_x V^n\|^2) - \frac{(\varDelta x)^4}{360}(\nabla_x(\eta^{n+1} - \eta^n), \nabla_x(V^{n+1}$$
$$+ V^n))] = -\frac{1}{2}(S - PR)(\nabla_x(\psi^{n+1} + \psi^n), \nabla_x(V^{n+1} + V^n)) - \frac{1}{2}P\|\nabla_x(V^{n+1} + V^n)\|^2.$$

(30)

Multiplying Eq. (28) by $\delta_x^2(\psi_i^{n+1} + \psi_i^n)$ and summing i from 1 to $N-1$ and using Lemma 1, yield

$$\frac{S}{\varDelta t}(\|\nabla_x\psi^{n+1}\|^2 - \|\nabla_x\psi^n\|^2) = -\frac{R}{2}\|\nabla_x(\psi^{n+1} + \psi^n)\|^2 + \frac{1}{2}(\delta_x(\psi^{n+1} + \psi^n), \delta_x(V^{n+1} + V^n)).$$

(31)

Multiplying Eq. (31) by *(S - PR)* and adding it to Eq.(30), we have

$$\frac{S}{\varDelta t}[\|V^{n+1}\|^2 - \|V^n\|^2 - \frac{(\varDelta x)^2}{12}(\|\nabla_x V^{n+1}\|^2 - \|\nabla_x V^n\|^2) - \frac{(\varDelta x)^4}{360}(\nabla_x(\eta^{n+1} - \eta^n), \nabla_x(V^{n+1}$$
$$+ V^n)) + (S - PR)(\|\nabla_x\psi^{n+1}\|^2 - \|\nabla_x\psi^n\|^2)] + \frac{R}{2}(S - PR)\|\nabla_x(\psi^{n+1} + \psi^n)\|^2$$
$$+ \frac{1}{2}P(\|\nabla_x(V^{n+1} + V^n)\|^2) = 0.$$

(32)

Multiplying Eq. (29) by $(\eta_i^{n+1} - \eta_i^n)$ and summing i from 1 to $N-1$ and using Lemma 1, yield

$$\|\eta^{n+1}\|^2 - \|\eta^n\|^2 = -(\nabla_x(\eta^{n+1} - \eta^n), \nabla_x(V^{n+1} + V^n)).$$

(33)

Multiplying Eq. (33) by $\frac{S(\varDelta x)^4}{\varDelta t \, 360}$ and adding it to Eq.(32), we have

$$\frac{S}{\Delta t}[\|V^{n+1}\|^2 - \|V^n\|^2 - \frac{\Delta x^2}{12}(\|\nabla_x V^{n+1}\|^2 - \|\nabla_x V^n\|^2) + \frac{\Delta x^4}{360}(\|\eta^{n+1}\|^2 - \|\eta^n\|^2)) +$$

$$(S-PR)(\|\nabla_x \psi^{n+1}\|^2 - \|\nabla_x \psi^n\|^2)] + \frac{R}{2}(S-PR)\|\nabla_x(\psi^{n+1}+\psi^n)\|^2$$

$$+\frac{1}{2}P(\|\nabla_x(V^{n+1}+V^n)\|^2) = 0.$$

Since R and P are positive constants and $S - PR > 0$, it follows that

$$\frac{S}{\Delta t}[\|V^{n+1}\|^2 - \|V^n\|^2 - \frac{\Delta x^2}{12}(\|\nabla_x V^{n+1}\|^2 - \|\nabla_x V^n\|^2) + \frac{\Delta x^4}{360}(\|\eta^{n+1}\|^2 - \|\eta^n\|^2)) +$$

$$(S-PR)(\|\nabla_x \psi^{n+1}\|^2 - \|\nabla_x \psi^n\|^2)] \le 0.$$

This is equivalent to

$$G(V^{n+1}) + \frac{\Delta x^4}{360}\|\eta_i^{n+1}\|^2 + (S-PR)\|\nabla_x \psi^{n+1}\|^2 \le G(V^n) + \frac{\Delta x^4}{360}\|\eta^n\|^2 + (S-PR)\|\nabla_x \psi^n\|^2.$$

(34)

by using the definition (23), (25) follows by recursion with n in (43) and (24). □

4.Numerical Results

In this section, we will test the accuracy of sixth order scheme and compare its results with the fourth and second order schemes We use the one dimensional test problem for microscale heat transport equation with initial and boundary conditions satisfying the exact solution $T(x,t) = e^{x+t}$, $0 \le t \le 1$, $0 \le x \le 1$. The code was written in Fortran power station 90 programming language. The average absolute error evaluated by using the following equation

$$Average\ absolute\ error = \frac{\sum_{i=1}^{N_x-1} |T_{ei} - T_i|}{(N_x - 1)},$$

where T_i represents the approximate value and T_{ei} represents the exact value.

Firstly, we chose $S = 2.5$, $P = 2$, $R = 0.5$, which correspond to the case $S - PR > 0$. The errors of the sixth order and the fourth order schemes are compared in Fig. 1 for $\Delta t = 0.001$ and two choices $\Delta x = 0.05$ and $\Delta x = 0.01$. The errors of the sixth order scheme are shown to be smaller than those of the fourth order scheme in both cases. Note that the truncation error is of order $O(\Delta t^2, \Delta x^4)$ for the fourth order scheme and of order $O(\Delta t^2, \Delta x^6)$ for the sixth order scheme. Thus, if Δt is large and the temporal error component dominates, the difference in error magnitude between the sixth order scheme and the forth order scheme will decrease.
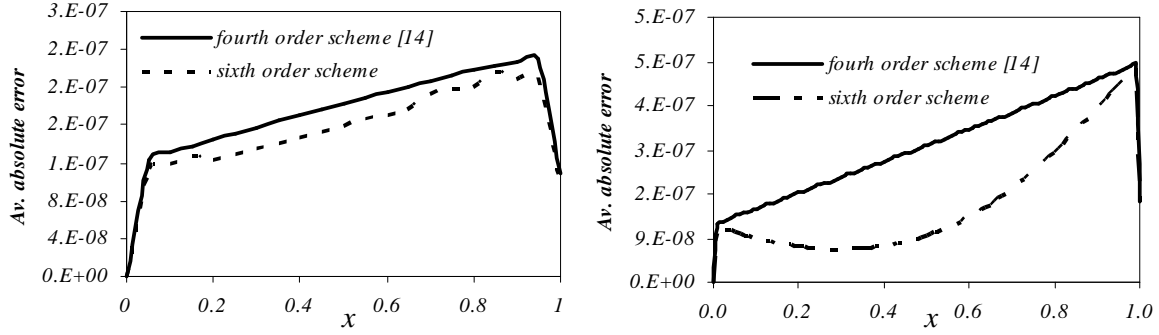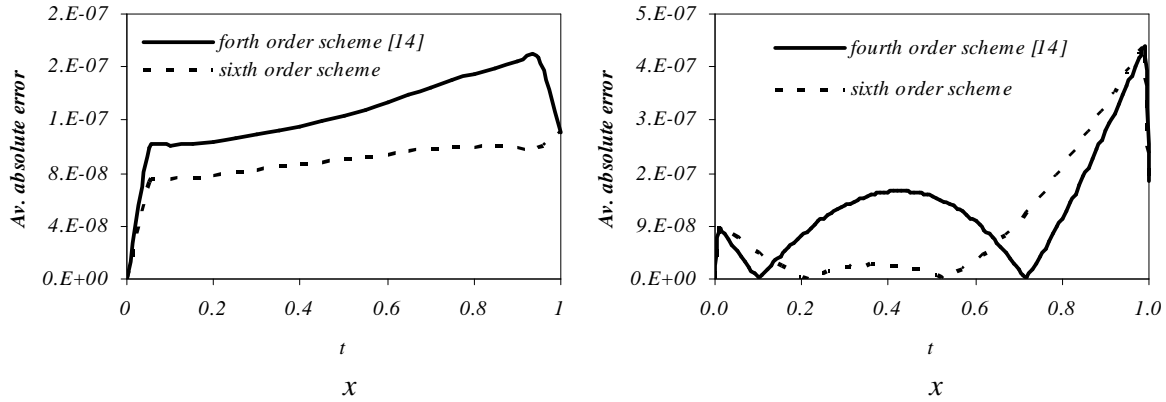
Fig. 1: Av. absolute error comparison of fourth and sixth order schemes with $S - PR > 0$, **at** $t = 1$ **and** $\Delta t = 0.001$ **(first test).left subfigure:** $\Delta x = 0.05$ **, right subfigure** $\Delta x = 0.01$

Fig. 2: Av. absolute error comparison of fourth and sixth order schemes with $S - PR = 0$, **at** $t = 1$ **and** $\Delta t = 0.001$ **(second test).left subfigure:** $\Delta x = 0.05$ **, right subfigure** $\Delta x = 0.01$

For the second test, we chose $S = 1$, $P = 1$, $R = 1$, This corresponds to $S - PR = 0$. The test results are plotted in Fig. 2. We see that the Av. absolute errors of the sixth order scheme much better than the fourth order scheme.

For the third test, we chose $S = 0.5$, $P = 1$, $R = 1.5$, This corresponds to $S - PR < 0$. The Av. absolute error against the time ($\Delta t = 0.001$) with $\Delta x = 0.05$ and $\Delta x = 0.01$ are graphed in Fig. 3.
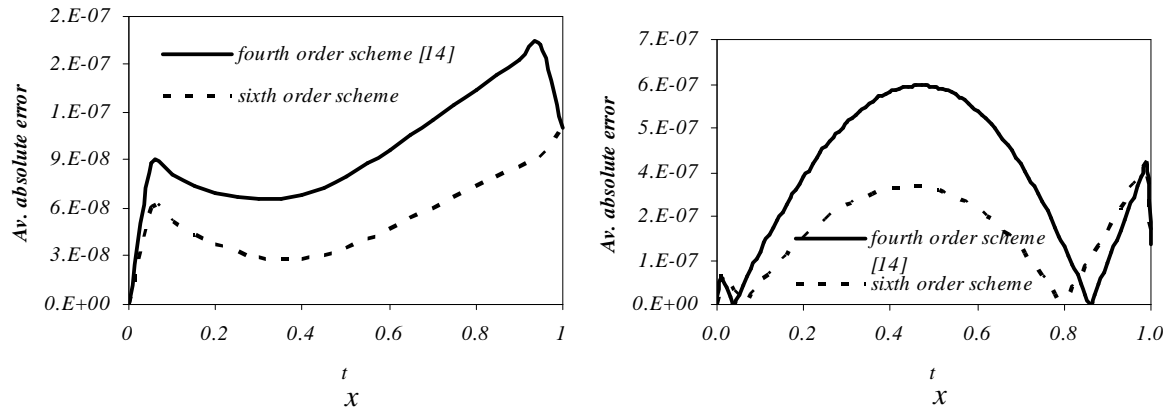
Fig.3: Av. absolute error comparison of fourth and sixth order schemes with
$S - PR < 0$, **at** $t = 1$ **and** $\Delta t = 0.001$ **(third test).left subfigure:** $\Delta x = 0.05$, **right subfigure** $\Delta x = 0.01$

The results of Av. absolute errors for T_i^n at $t = 1.0$ are listed in table 1. Computations are carried out for $(\Delta x = 0.05, \Delta x = 0.01)$ and $(\Delta t = 0.002, \Delta t = 0.001)$ using the second, the fourth order and the sixth order finite difference schemes. Also, we note that the errors of the sixth order scheme are to be smaller than those of the second and fourth order schemes.

Table 1: The absolute errors for $T(x,1)$ with three test problems

	Test1		Test2		Test3	
$\Delta x = 0.05$	$\Delta t = 0.002$	$\Delta t = 0.001$	$\Delta t = 0.002$	$\Delta t = 0.001$	$\Delta t = 0.002$	$\Delta t = 0.001$
Second order	5.69E-05	5.62E-05	6.42E-05	6.37E-05	6.41E-05	6.38E-05
Fourth order	8.54E-07	1.81E-07	6.93E-07	1.28E-07	6.23E-07	9.78E-08
Sixth order	1.65E-07	7.40E-08	6.48E-08	3.79E-08	6.43E-08	3.83E-08
$\Delta x = 0.01$						
Second order	3.07E-06	2.33E-06	3.00E-06	2.44E-06	2.42E-06	2.09E-06
Fourth order	8.32E-07	1.60E-07	5.55E-07	1.19E-07	2.92E-07	3.52E-07
Sixth order	2.43E-07	8.76E-08	1.80E-07	9.77E-08	1.59E-07	8.17E-08

Now, if we use the test problem of [14] which satisfying the exact solution $T(x,t) = e^{-\pi^2 t} \sin \pi x$ for $t \geq 0$, $x \in [0,1]$. We chose $S = \dfrac{3}{2\pi^2}$, $P = \dfrac{2}{\pi^2}$, $R = 0.5$, which correspond to the case $S - PR > 0$, The errors of the sixth order and the fourth order schemes are compared in Fig. 4 for $\Delta t = 0.001$ and two choices $\Delta x = 0.05$ and $\Delta x = 0.01$.
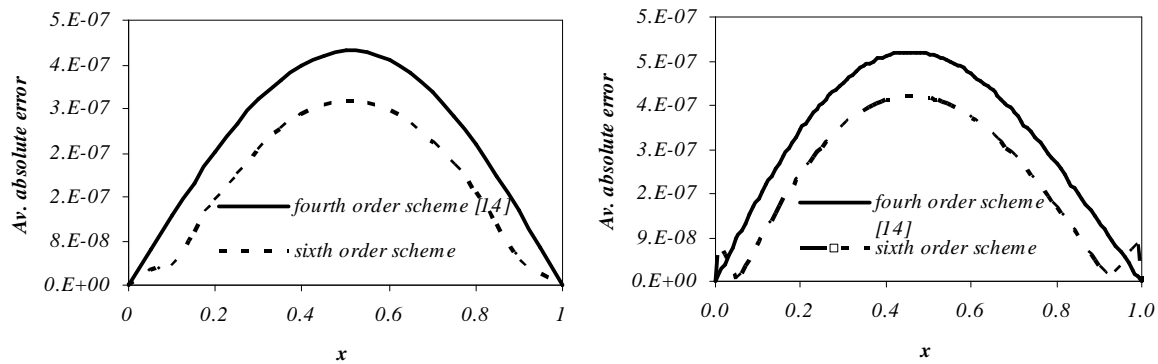
Fig. 4: Av. absolute error comparison of fourth and sixth order schemes with $S - PR > 0$, **at** $t = 1$ **and** $\Delta t = 0.001$ **(test problem of [14]).left subfigure:** $\Delta x = 0.05$ **, right subfigure** $\Delta x = 0.01$

5.Conclusions

We have introduced a sixth order compact finite difference scheme with Crank-Nicholson technique for solving one dimensional microscale heat transport equation. Our scheme is unconditionally stable with respect to the initial values. Our Numerical results showed that the sixth order compact scheme is computationally more efficient and more accurate than the second and fourth order scheme.

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اسلوب الفروقات المحددة المضغوطة من رتب عليا لحل معادلة التوصيل الحراري احادية البعد

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المستخلص

تناولنا في هذا البحث حل معادلة التوصيل الحراري احادية البعد باستعمال اسلوب الفروقات المحددة المضغوطة من الرتبة السادسة بالنسبة للحيز مع تطبيق طريقة كرانك نيكولسون بالنسبة للزمن. وقد اثبتنا ان هذا الاسوب مستقر بالنسبة لشروطه الابتدائية. وقد قمنا بمقارنة دقة النتائج العددية لصيغة الفروقات المحددة من الرتبة السادسة مع الفروقات المحددة من الرتبة الرابعة والثانية. ووجدنا ان النتائج العددية لاسلوب الفروقات المحددة المضغوطة من الرتبة السادسة اكثر دقة من اساليب الفروقات المحددة المضغوطة من الرتبة الرابعة والثانية.