Fourth Order Compact Finite Difference Scheme for Two-dimensional Microscale Heat Equation.

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Abstract

In this paper, fourth order compact difference scheme is employed to solve two dimensional microscale heat transport equation. By introducing an intermediate function for the heat transport equation, we use the fourth order compact scheme for the space with a second order Crank-Nicolson scheme for the time. The stability of this scheme is proved unconditionally stable with respect to initial values. The computational accuracy is demonstrated that the results of the compact fourth order finite difference scheme is more accurate than the second order finite difference scheme [12].

Key words: heat transport equation, finite differences, fourth order compact, Crank-Nicolson scheme.

1. Introduction

The two dimension Microscale heat transport equation for describing the thermal behavior of thin films and other microstructure can be written as [12]

$$\frac{1}{\alpha}\left(\frac{\partial T}{\partial t}+T_q\frac{\partial^2 T}{\partial t^2}\right)=T_q\frac{\partial^3 T}{\partial t\,\partial x^2}+T_T\frac{\partial^3 T}{\partial t\,\partial y^2}+\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+S\,, \tag{1}$$

with the initial conditions :

$$T(x,y,0)=T_0(x,y), \qquad \frac{\partial T(x,y,0)}{\partial t}=T_1(x,y), \tag{2}$$

and the boundary conditions:

$$\begin{aligned} &T(0,y,t)=T_2(y,t), && T(x,0,t)=T_3(x,t),\\ &T(L_x,y,t)=T_4(x,t), && T(x,L_y,t)=T_5(x,t), \end{aligned} \tag{3}$$

where T is the temperature, α, T_T and T_q are positive constants. Here α is the thermal diffusivity. T_T and T_q represent the time lags of heat flux and temperature gradient, respectively. S is the sours [11].

Many applications, can be modeled by the microscale heat transport equation for examples, phonon electron interaction model [7], the single energy equation [9, 10], the phonon scatting model [2], the phonon radiative transfer model [3] and the lagging behavior model[5, 9, 8]. Few authors deal with numerical solution of one dimension microscale heat transport equation. By using Crank-Nicolson technique Qui and Tien [6] have solved the phonon electron interaction model. Joshi and Majumdar [3] have used the explicit upstream difference method to solve the phonon radiative transfer model in one dimensional medium. Zhang and Zhao [11] have solved the one dimension microscale heat transport equation using fourth order compact scheme and have proved this new scheme unconditionally with initial values.

Only Zhang and Zhao in [12] and [13] solve the two and three dimension microscale heat transport equation, respectively, using second order for time and space.

In this paper we consider the microscale heat transport equation in two dimensions. The aim of this paper is to develop the work of Zhang and Zhao [12] for $T_T = T_q$. We increase the accuracy of the finite difference scheme by using a fourth order accuracy for space and second order for time, while Zhang and Zhao [12] using a second order for space and second for time using Crank Nicolson technique . In addition, we prove that the fourth order compact scheme is unconditionally stable with initial values. Numerical results are provided for comparison and testing purpose in [12] by Zhang and Zhao.

2. Fourth order compact discretization

For convenience, let us consider a rectangular domain $\Omega = [0, L_x] \times [0, L_y]$. Here subscripts are obviously not derivatives. We discretize Ω with uniform mesh sizes Δx and Δy respectively in the x and y coordinate directions. Define $Nx = Lx/\Delta x$ and $Ny = Ly/\Delta y$ the numbers of uniform subintervals along the x and y coordinate directions, respectively. The mesh points are (x_i, y_j) where $x_i = i\,\Delta x$ and $y_j = j\,\Delta y,\ 0 \le i \le Nx,\ 0 \le j \le Ny$. In the sequel, we may also use the index pair (i, j) to represent the mesh point (x_i, y_j). Also, we discretize the time interval with uniform mesh sizes Δt.

If $T_T = T_q$ then equation (1) can be written as;

$$\frac{1}{\alpha}\left(\frac{\partial T}{\partial t} + T_q \frac{\partial^2 T}{\partial t^2}\right) = T_q \frac{\partial^3 T}{\partial t\, \partial x^2} + T_q \frac{\partial^3 T}{\partial t\, \partial y^2} + \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + S\,. \tag{4}$$

Consider the following function

$$\theta = T + T_q \frac{\partial T}{\partial t}\,. \tag{5}$$

Substituting (5) into (4), and after simplification, we get

$$\frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} = f\,, \tag{6}$$

where $f = \frac{1}{\alpha}\frac{\partial \theta}{\partial t} - S$, with the modified initial and boundary conditions

$$\left.\begin{aligned}
&\theta(x,y,0) = T_0(x,y) + T_q T_1(x,y), \\
&\theta(0,y,t) = T_2(y,t) + T_q \frac{\partial T_2(y,t)}{\partial t}, \qquad \theta(x,0,t) = T_3(x,t) + T_q \frac{\partial T_3(x,t)}{\partial t}, \\
&\theta(L_x,y,t) = T_4(y,t) + T_q \frac{\partial T_4(y,t)}{\partial t}, \qquad \theta(x,L_y,t) = T_5(x,t) + T_q \frac{\partial T_5(x,t)}{\partial t}.
\end{aligned}\right\} \tag{7}$$

The standard second order central difference operators at grid point (i, j) can be written as

$$\left.\begin{aligned}
\left.\frac{\partial^2 \theta}{\partial x^2}\right|_{ij} &= \delta_x^2 \theta_{ij} - \frac{\Delta x^2}{12}\frac{\partial^4 \theta}{\partial x^4} - \frac{\Delta x^4}{360}\frac{\partial^6 \theta}{\partial x^6} + O(\Delta x^6), \\
\left.\frac{\partial^2 \theta}{\partial y^2}\right|_{ij} &= \delta_y^2 \theta_{ij} - \frac{\Delta y^2}{12}\frac{\partial^4 \theta}{\partial y^4} - \frac{\Delta y^4}{360}\frac{\partial^6 \theta}{\partial y^6} + O(\Delta y^6).
\end{aligned}\right\} \tag{8}$$

By using these finite difference approximations, Eq.(6) can be discretized at a given grid point (i, j) as

$$\delta_x^2\theta_{ij} + \delta_y^2\theta_{ij} = f_{ij} + O(\Delta x^2, \Delta y^2). \quad (9)$$

The fourth order compact scheme for (6) can be introduced as [14]

$$\delta_x^2\theta_{ij} + \delta_y^2\theta_{ij} + \frac{1}{12}(\Delta x^2 + \Delta y^2)\delta_x^2\delta_y^2\theta_{ij} = f_{ij} + \frac{\Delta x^2}{12}\delta_x^2 f_{ij} + \frac{\Delta y^2}{12}\delta_y^2 f_{ij} + O(\Delta x^4 + \Delta y^4). \quad (10)$$

Applying the Crank-Nicolson technique to the fourth order compact scheme (10) where ($f = \frac{1}{\alpha}\frac{\partial\theta}{\partial t} - S$), we get

$$\frac{1}{2}\delta_x^2(\theta_{ij}^{n+1} + \theta_{ij}^n) + \frac{1}{2}\delta_y^2(\theta_{ij}^{n+1} + \theta_{ij}^n) + \frac{1}{24}(\Delta x^2 + \Delta y^2)\delta_x^2\delta_y^2(\theta_{ij}^{n+1} + \theta_{ij}^n) = [\frac{1}{\alpha}\frac{\partial\theta}{\partial t} - S^{n+\frac{1}{2}}$$
$$+ \frac{\Delta x^2}{12}\delta_x^2(\frac{1}{\alpha}\frac{\partial\theta}{\partial t} - S^{n+\frac{1}{2}}) + \frac{\Delta y^2}{12}\delta_y^2(\frac{1}{\alpha}\frac{\partial\theta}{\partial t} - S^{n+\frac{1}{2}})]_{ij}, \quad (11)$$

with $\left.\frac{\partial\theta}{\partial t}\right|_{ij} = \frac{\theta_{ij}^{n+1} - \theta_{ij}^n}{\Delta t}$.

Then the final formula can be written as follows:

$$\frac{1}{2}\delta_x^2(\theta_{ij}^{n+1} + \theta_{ij}^n) + \frac{1}{2}\delta_y^2(\theta_{ij}^{n+1} + \theta_{ij}^n) + \frac{1}{24}(\Delta x^2 + \Delta y^2)\delta_x^2\delta_y^2(\theta_{ij}^{n+1} + \theta_{ij}^n) = \frac{1}{\alpha}(\frac{\theta_{ij}^{n+1} - \theta_{ij}^n}{\Delta t})$$
$$- S_{ij}^{n+\frac{1}{2}} + \frac{\Delta x^2}{12}\delta_x^2(\frac{1}{\alpha}(\frac{\theta_{ij}^{n+1} - \theta_{ij}^n}{\Delta t}) - S_{ij}^{n+\frac{1}{2}}) + \frac{\Delta y^2}{12}\delta_y^2(\frac{1}{\alpha}(\frac{\theta_{ij}^{n+1} - \theta_{ij}^n}{\Delta t}) - S_{ij}^{n+\frac{1}{2}}), \quad (12)$$

where

$$\delta_x^2\delta_y^2\theta_{ij} = \frac{1}{(\Delta x\Delta y)^2}\left[4\theta_{ij} - 2(\theta_{i-1j} + \theta_{i+1j} + \theta_{ij-1} + \theta_{ij+1}) + \theta_{i-1j-1} + \theta_{i+1j-1} + \theta_{i+1j+1} + \theta_{i-1j+1}\right]$$

After simplification (12), we obtain

$$a\theta_{ij}^{n+1} + b(\theta_{i+1j}^{n+1} + \theta_{i-1j}^{n+1}) + c(\theta_{ij-1}^{n+1} + \theta_{ij+1}^{n+1}) + d(\theta_{i+1j-1}^{n+1} + \theta_{i-1j-1}^{n+1} + \theta_{i+1j+1}^{n+1} + \theta_{i-1j+1}^{n+1}) = F_{ij}^n, \quad (13)$$

where

$$F_{ij}^n = a_1\theta_{ij}^n + b_1(\theta_{i+1j}^n + \theta_{i-1j}^n) + c_1(\theta_{ij-1}^n + \theta_{ij+1}^n) + d_1(\theta_{i+1j-1}^n + \theta_{i-1j-1}^n + \theta_{i+1j+1}^n$$
$$+ \theta_{i-1j+1}^n) + R(S_{ij}^{n+\frac{1}{2}}),$$

$$a = \frac{2}{3\alpha\Delta t} + \frac{5}{6}(\frac{1}{\Delta x^2} + \frac{1}{\Delta y^2}),$$

$$b = \frac{2}{12\alpha\Delta t} - \frac{5}{12}\frac{1}{\Delta x^2} + \frac{1}{12}\frac{1}{\Delta y^2},$$

$$c = \frac{2}{12\alpha\Delta t} - \frac{5}{12}\frac{1}{\Delta y^2} + \frac{1}{12}\frac{1}{\Delta x^2},$$

$$d = -\frac{1}{24}(\frac{1}{\Delta y^2} + \frac{1}{\Delta x^2}),$$

$$a_1=\frac{2}{3\alpha\Delta t}-\frac{5}{6}(\frac{1}{\Delta x^2}+\frac{1}{\Delta y^2}),$$

$$b_1=\frac{2}{12\alpha\Delta t}+\frac{5}{12}\frac{1}{\Delta x^2}-\frac{1}{12}\frac{1}{\Delta y^2},$$

$$c_1=\frac{2}{12\alpha\Delta t}+\frac{5}{12}\frac{1}{\Delta y^2}-\frac{1}{12}\frac{1}{\Delta x^2},$$

$$d_1=\frac{1}{24}(\frac{1}{\Delta y^2}+\frac{1}{\Delta x^2}),$$

$$R(S_{ij}^{n+\frac{1}{2}})=S_{ij}^{n+\frac{1}{2}}+\frac{\Delta x^2}{12}\delta_x^2 S_{ij}^{n+\frac{1}{2}}+\frac{\Delta y^2}{12}\delta_y^2 S_{ij}^{n+\frac{1}{2}}.$$

Equation (13) used to evaluate θ_{ij}^{n+1}. For computing T_{ij}^{n+1}, we discretize (5) using Crank-Nicolson method

$$\frac{1}{2}(\theta_{ij}^{n+1}+\theta_{ij}^{n})=\frac{1}{2}(T_{ij}^{n+1}+T_{ij}^{n})+\frac{T_q}{\Delta t}(T_{ij}^{n+1}-T_{ij}^{n}). \quad (14)$$

Simplification equation (14) to get T_{ij}^{n+1}, we have

$$T_{ij}^{n+1}=(\frac{\Delta t}{2}+T_q)^{-1}(T_q-\frac{\Delta t}{2})T_{ij}^{n}+(\frac{\Delta t}{2}+T_q)^{-1}\frac{\Delta t}{2}(\theta_{ij}^{n+1}+\theta_{ij}^{n}).$$

which can be used to evaluate T_{ij}^{n+1}.

3. Stability analysis

In this section, we prove that the fourth order compact scheme (12) and (14) is unconditionally stable with respect to the initial values. To prove this, we use the discrete energy method [1, 5]. For achieving this purpose, we will define D as the set of discrete values

$$\{e^n=\{e_{ij}^n\},\ with\ e_{0j}^n=e_{Nj}^n=e_{i0}^n=e_{iN}^n=0,\ 1\le i,j\le N\}.$$

We then make the following norm definitions for any $e^n,\ f^n\in D$,

$$(e^n,f^n)=\Delta x^2\sum_{i,j=1}^{n}e_{ij}^n f_{ij}^n, \qquad \|e^n\|^2=(e^n,e^n).$$

The following results can be verified easily [1, 4].

Lemma 1: For any $e^n,\ f^n\in D$, the following equalities hold.

$$(\delta_x^2 e^n,f^n)=-(\delta_x e^n,\delta_x f^n), \quad (\delta_y^2 e^n,f^n)=-(\delta_y e^n,\delta_y f^n),$$

$$(\delta_x^2\delta_y^2 e^n,f^n)=-(\delta_x\delta_y e^n,\delta_x\delta_y f^n),$$

where $\delta_x e^n=\frac{e_{i+1j}^n-e_{ij}^n}{\Delta x}$, $\qquad \delta_y e^n=\frac{e_{ij+1}^n-e_{ij}^n}{\Delta y}$.

are the forward difference operators.

Theorem: Suppose that $\{T_{ij}^n,\theta_{ij}^n\}$ and $\{V_{ij}^n,\xi_{ij}^n\}$ are the solution of the finite difference scheme (12) and (14) which satisfy the boundary conditions (3) and (7), and have different initial values $\{T_{ij}^0, \theta_{ij}^0\}$ and $\{V_{ij}^0, \xi_{ij}^0\}$, respectively. Let, $w_{ij}^n = \theta_{ij}^n - \xi_{ij}^n$ and $\varepsilon_{ij}^n = T_{ij}^n - V_{ij}^n$ then $\{w_{ij}^n, \varepsilon_{ij}^n\}$ satisfy

$$\frac{1}{\alpha}\left[\left\|w^n\right\|^2 + \frac{\Delta x^2}{12}\left\|\delta_x w^n\right\|^2 + \frac{\Delta y^2}{12}\left\|\delta_y w^n\right\|^2\right] + 2T_q\left\|\delta_x \varepsilon^n\right\|^2 + 2T_q\left\|\delta_y \varepsilon^n\right\|^2$$
$$+\frac{1}{6}\left(\Delta x^2 + \Delta y^2\right)\frac{T_q}{\Delta t}\left\|\delta_x\delta_y \varepsilon^n\right\|^2 \le \frac{1}{\alpha}\left[\left\|w^0\right\|^2 + \frac{\Delta x^2}{12}\left\|\delta_x w^0\right\|^2 + \frac{\Delta y^2}{12}\left\|\delta_y w^0\right\|^2\right]$$
$$+2T_q\left\|\delta_x \varepsilon^0\right\|^2 + 2T_q\left\|\delta_y \varepsilon^0\right\|^2 + \frac{1}{6}\left(\Delta x^2 + \Delta y^2\right)\frac{T_q}{\Delta t}\left\|\delta_x\delta_y \varepsilon^0\right\|^2. \tag{15}$$

for any $0 \le n\,\Delta t \le t_{stop}$. This implies that the finite difference scheme is unconditionally stable with respect to the initial values.

Proof: Firstly, we substitute (14) into (12) we get,

$$\frac{1}{2}\delta_x^2(T_{ij}^{n+1} + T_{ij}^n) + \frac{T_q}{\Delta t}\delta_x^2(T_{ij}^{n+1} - T_{ij}^n) + \frac{1}{2}\delta_y^2(T_{ij}^{n+1} + T_{ij}^n) + \frac{T_q}{\Delta t}\delta_y^2(T_{ij}^{n+1} - T_{ij}^n)$$
$$+\frac{1}{24}(\Delta x^2 + \Delta y^2)\delta_x^2\delta_y^2(T_{ij}^{n+1} + T_{ij}^n) + \frac{T_q}{12\Delta t}(\Delta x^2 + \Delta y^2)\delta_x^2\delta_y^2(T_{ij}^{n+1} - T_{ij}^n) \tag{16}$$
$$= \frac{1}{\alpha\Delta t}\left[\theta_{ij}^{n+1} - \theta_{ij}^n + \frac{\Delta x^2}{12}\delta_x^2(\theta_{ij}^{n+1} - \theta_{ij}^n) + \frac{\Delta y^2}{12}\delta_y^2(\theta_{ij}^{n+1} - \theta_{ij}^n)\right].$$

Since $\{T_{ij}^n, \theta_{ij}^n\}$ and $\{V_{ij}^n, \xi_{ij}^n\}$ are solutions of (16) with the same boundary conditions, so $\{w_{ij}^n, \varepsilon_{ij}^n\} \in D$, and they also satisfy

$$\frac{1}{\alpha\,\Delta t}\left[w_{ij}^{n+1} - w_{ij}^n + \frac{\Delta x^2}{12}\delta_x^2(w_{ij}^{n+1} - w_{ij}^n) + \frac{\Delta y^2}{12}\delta_y^2(w_{ij}^{n+1} - w_{ij}^n)\right]$$
$$= \frac{1}{2}\left(\delta_x^2\left(\varepsilon_{ij}^{n+1} + \varepsilon_{ij}^n\right)\right) + \frac{1}{2}\left(\delta_y^2\left(\varepsilon_{ij}^{n+1} + \varepsilon_{ij}^n\right)\right)$$
$$+\frac{T_q}{\Delta t}\left(\delta_x^2\left(\varepsilon_{ij}^{n+1} - \varepsilon_{ij}^n\right)\right) + \frac{T_q}{\Delta t}\left(\delta_y^2\left(\varepsilon_{ij}^{n+1} - \varepsilon_{ij}^n\right)\right)$$
$$+\frac{1}{24}\left(\Delta x^2 + \Delta y^2\right)\delta_x^2\delta_y^2\left(\varepsilon_{ij}^{n+1} + \varepsilon_{ij}^n\right) + \frac{T_q}{12\Delta t}\left(\Delta x^2 + \Delta y^2\right)\delta_x^2\delta_y^2\left(\varepsilon_{ij}^{n+1} - \varepsilon_{ij}^n\right). \tag{17}$$

From (14), we can see that

$$w_{ij}^{n+1} + w_{ij}^n = \left(\varepsilon_{ij}^{n+1} + \varepsilon_{ij}^n\right) + \frac{2T_q}{\Delta t}\left(\varepsilon_{ij}^{n+1} - \varepsilon_{ij}^n\right). \tag{18}$$

Using (18) and Lemma 1, we can obtain the following equalities:

$$\left(\delta_x\left(\varepsilon^{n+1}+\varepsilon^n\right),\ \delta_x\left(w^{n+1}+w^n\right)\right)=\left\|\delta_x\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\varepsilon^{n+1}\right\|^2-\left\|\delta_x\varepsilon^n\right\|^2\right),$$

$$\left(\delta_x\left(\varepsilon^{n+1}-\varepsilon^n\right),\ \delta_x\left(w^{n+1}+w^n\right)\right)=\left\|\delta_x\varepsilon^{n+1}\right\|^2-\left\|\delta_x\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right),$$

$$\left(\delta_y\left(\varepsilon^{n+1}+\varepsilon^n\right),\ \delta_y\left(w^{n+1}+w^n\right)\right)=\left\|\delta_y\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_y\varepsilon^n\right\|^2\right),$$

$$\left(\delta_y\left(\varepsilon^{n+1}-\varepsilon^n\right),\ \delta_y\left(w^{n+1}+w^n\right)\right)=\left\|\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_y\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_y\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right),$$

$$\left(\delta_x\delta_y\left(\varepsilon^{n+1}+\varepsilon^n\right),\ \delta_x\delta_y\left(w^{n+1}+w^n\right)\right)=\left\|\delta_x\delta_y\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_x\delta_y\varepsilon^n\right\|^2\right),$$

$$\left(\delta_x\delta_y\left(\varepsilon^{n+1}-\varepsilon^n\right),\ \delta_x\delta_y\left(w^{n+1}+w^n\right)\right)=\left\|\delta_x\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_x\delta_y\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\delta_y\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right),$$

(19)

By multiplying both sides of (17) by $\left(w_{ij}^{n+1}+w_{ij}^n\right)\Delta x\Delta y$ and sum over i and j we can get

$$\frac{1}{\alpha\,\Delta t}\left[\left(\left\|w^{n+1}\right\|^2-\left\|w^n\right\|^2\right)+\frac{\Delta x^2}{12}\left(\left\|\delta_x w^{n+1}\right\|^2-\left\|\delta_x w^n\right\|^2\right)+\frac{\Delta y^2}{12}\left(\left\|\delta_y w^{n+1}\right\|^2-\left\|\delta_y w^n\right\|^2\right)\right]$$
$$=\frac{1}{2}\left(\delta_x^2\left(\varepsilon^{n+1}+\varepsilon^n\right),\ w^{n+1}+w^n\right)+\frac{1}{2}\left(\delta_y^2\left(\varepsilon^{n+1}+\varepsilon^n\right),\ w^{n+1}+w^n\right)$$
$$+\frac{T_q}{\Delta t}\left(\delta_x^2\left(\varepsilon^{n+1}-\varepsilon^n\right),\ w^{n+1}+w^n\right)+\frac{Tq}{\Delta t}\left(\delta_y^2\left(\varepsilon^{n+1}-\varepsilon^n\right),\ w^{n+1}+w^n\right)$$
$$+\frac{1}{24}\left(\Delta x^2+\Delta y^2\right)\left(\delta_x^2\delta_y^2\left(\varepsilon^{n+1}+\varepsilon^n\right),w^{n+1}+w^n\right)$$
$$+\frac{T_q}{12\Delta t}\left(\Delta x^2+\Delta y^2\right)\left(\delta_x^2\delta_y^2\left(\varepsilon^{n+1}-\varepsilon^n\right),w^{n+1}+w^n\right).$$

(20)

We can find each term on the right-hand side of (20). Using lemma 1 and (19), we can get

$$\frac{1}{2}\left(\delta_x^2\left(\varepsilon^{n+1}+\varepsilon^n\right),\ w^{n+1}+w^n\right)=-\frac{1}{2}\left(\left\|\delta_x\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\varepsilon^{n+1}\right\|^2-\left\|\delta_x\varepsilon^n\right\|^2\right)\right),$$

$$\frac{T_q}{\Delta t}\left(\delta_x^2\left(\varepsilon^{n+1}-\varepsilon^n\right),\ w^{n+1}+w^n\right)=-\frac{T_q}{\Delta t}\left(\left\|\delta_x\varepsilon^{n+1}\right\|^2-\left\|\delta_x\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right)\right),$$

$$\frac{1}{2}\left(\delta_y^2\left(\varepsilon^{n+1}+\varepsilon^n\right),\ w^{n+1}+w^n\right)=-\frac{1}{2}\left(\left\|\delta_y\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_y\varepsilon^n\right\|^2\right)\right),$$

$$\frac{T_q}{\Delta t}\left(\delta_y^2\left(\varepsilon^{n+1}-\varepsilon^n\right),\ w^{n+1}+w^n\right)=-\frac{T_q}{\Delta t}\left(\left\|\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_y\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_y\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right)\right),$$

$$\frac{1}{24}\left(\Delta x^2+\Delta y^2\right)\left(\delta_x^2\delta_y^2\left(\varepsilon^{n+1}+\varepsilon^n\right),w^{n+1}+w^n\right)$$
$$=-\frac{1}{24}\left(\Delta x^2+\Delta y^2\right)\left[\left\|\delta_x\delta_y\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_x\delta_y\varepsilon^n\right\|^2\right)\right],$$
$$\frac{T_q}{12\Delta t}\left(\Delta x^2+\Delta y^2\right)\left(\delta_x^2\delta_y^2\left(\varepsilon^{n+1}-\varepsilon^n\right),w^{n+1}+w^n\right)$$
$$=-\frac{T_q}{12\Delta t}\left(\Delta x^2+\Delta y^2\right)\left[\left\|\delta_x\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_x\delta_y\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\delta_y\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right)\right].$$
(21)

Substituting (21) into (20), yields

$$\frac{1}{\alpha\Delta t}\left[\left(\left\|w^{n+1}\right\|^2-\left\|w^n\right\|^2\right)+\frac{\Delta x^2}{12}\left(\left\|\delta_x w^{n+1}\right\|^2-\left\|\delta_x w^n\right\|^2\right)+\frac{\Delta y^2}{12}\left(\left\|\delta_y w^{n+1}\right\|^2-\left\|\delta_y w^n\right\|^2\right)\right]$$
$$=-\frac{1}{2}\left(\left\|\delta_x\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\varepsilon^{n+1}\right\|^2-\left\|\delta_x\varepsilon^n\right\|^2\right)\right)$$
$$-\frac{1}{2}\left(\left\|\delta_y\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_y\varepsilon^n\right\|^2\right)\right)$$
$$-\frac{T_q}{\Delta t}\left(\left\|\delta_x\varepsilon^{n+1}\right\|^2-\left\|\delta_x\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right)\right)$$
$$-\frac{T_q}{\Delta t}\left(\left\|\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_y\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_y\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right)\right)$$
$$-\frac{1}{24}\left(\Delta x^2+\Delta y^2\right)\left[\left\|\delta_x\delta_y\left(\varepsilon^{n+1}+\varepsilon^n\right)\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_x\delta_y\varepsilon^n\right\|^2\right)\right]$$
$$-\frac{T_q}{12\Delta t}\left(\Delta x^2+\Delta y^2\right)\left[\left\|\delta_x\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_x\delta_y\varepsilon^n\right\|^2+\frac{2T_q}{\Delta t}\left(\left\|\delta_x\delta_y\left(\varepsilon^{n+1}-\varepsilon^n\right)\right\|^2\right)\right].$$
(22)

after dropping the six negative terms from the right-hand side of (23), we get

$$\frac{1}{\alpha\Delta t}\left[\left(\left\|w^{n+1}\right\|^2-\left\|w^n\right\|^2\right)+\frac{\Delta x^2}{12}\left(\left\|\delta_x w^{n+1}\right\|^2-\left\|\delta_x w^n\right\|^2\right)+\frac{\Delta y^2}{12}\left(\left\|\delta_y w^{n+1}\right\|^2-\left\|\delta_y w^n\right\|^2\right)\right]$$
$$\le-\frac{2T_q}{\Delta t}\left\|\delta_x\varepsilon^{n+1}\right\|^2+\frac{2T_q}{\Delta t}\left\|\delta_x\varepsilon^n\right\|^2-\frac{2T_q}{\Delta t}\left\|\delta_y\varepsilon^{n+1}\right\|^2+\frac{2T_q}{\Delta t}\left\|\delta_y\varepsilon^n\right\|^2$$
$$-\frac{1}{6}\left(\Delta x^2+\Delta y^2\right)\frac{T_q}{\Delta t}\left(\left\|\delta_x\delta_y\varepsilon^{n+1}\right\|^2-\left\|\delta_x\delta_y\varepsilon^n\right\|^2\right).$$

This implies the following inequality

$$\frac{1}{\alpha}\left[\left\|w^{n+1}\right\|^2+\frac{\Delta x^2}{12}\left\|\delta_x w^{n+1}\right\|^2+\frac{\Delta y^2}{12}\left\|\delta_y w^{n+1}\right\|^2\right]+2T_q\left\|\delta_x\varepsilon^{n+1}\right\|^2+2T_q\left\|\delta_y\varepsilon^{n+1}\right\|^2$$
$$+\frac{1}{6}\left(\Delta x^2+\Delta y^2\right)\frac{T_q}{\Delta t}\left\|\delta_x\delta_y\varepsilon^{n+1}\right\|^2\le\frac{1}{\alpha}\left[\left\|w^n\right\|^2+\frac{\Delta x^2}{12}\left\|\delta_x w^n\right\|^2+\frac{\Delta y^2}{12}\left\|\delta_y w^n\right\|^2\right]$$
$$+2T_q\left\|\delta_x\varepsilon^n\right\|^2+2T_q\left\|\delta_y\varepsilon^n\right\|^2+\frac{1}{6}\left(\Delta x^2+\Delta y^2\right)\frac{T_q}{\Delta t}\left\|\delta_x\delta_y\varepsilon^n\right\|^2. \quad (23)$$

(15) follows from (23) by recursion with respect to n. □

4.Numerical Results

We consider two dimensional model problem to test the high order compact formulation for microscale heat transport equation with initial and boundary conditions satisfying the exact solution $T(x,y,t)=e^{x+y+t}$, $0\le t\le 1$, $0\le x,y\le 1$. The code was written in Fortran power station 90 programming language. The absolute error evaluated by using the following equation

$$Av.\left|Error\right|=\frac{\sum_{i=1}^{N_x-1}\sum_{j=1}^{N_y-1}\left|T_{eij}-T_{ij}\right|}{(N_x-1)(N_y-1)},$$

where T_{ij} represents the approximate value and T_{eij} represents the exact value.We chose $T_q=1,\ \alpha=0.5,\ S=0.0$,. The errors of the fourth order and the second order schemes are compared in Fig. 1 and Fig. 2 for $0.001\le\Delta t\le 0.02$ with two choices $\Delta x=\Delta y=0.1$ and $\Delta x=\Delta y=0.05$. The errors of the fourth order scheme are shown to be smaller than those of the second order scheme in both cases. Note that the truncation error is of order $O(\Delta t^2,\Delta x^4,\Delta y^4)$ for the fourth order scheme and of order $O(\Delta t^2,\Delta x^2,\Delta y^2)$ for the second order scheme. Thus, if Δt is large and the temporal error component dominates, the difference in error magnitude between the fourth order scheme and the second order scheme will decrease.

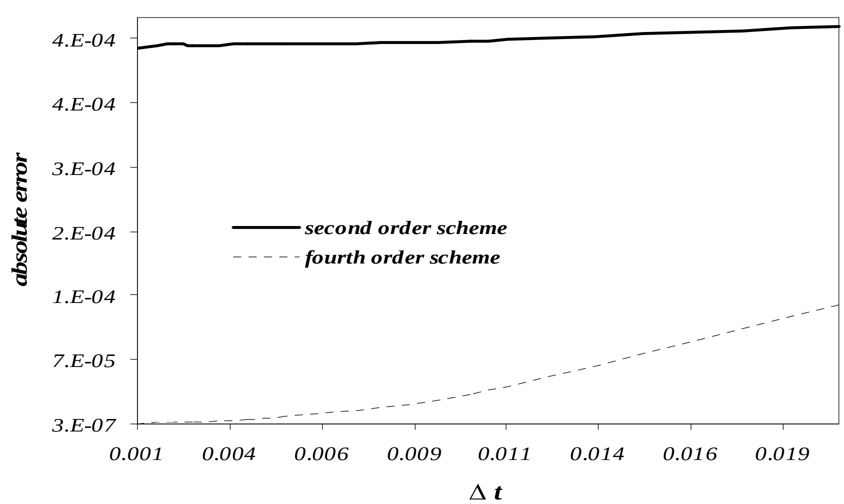

Fig. 1: Absolute error comparison of second and fourth order schemes at $T_q=1,\quad \alpha=0.5,\quad S=0.0$ **,** $\Delta x=\Delta y=0.1$ **and t =1.0.**

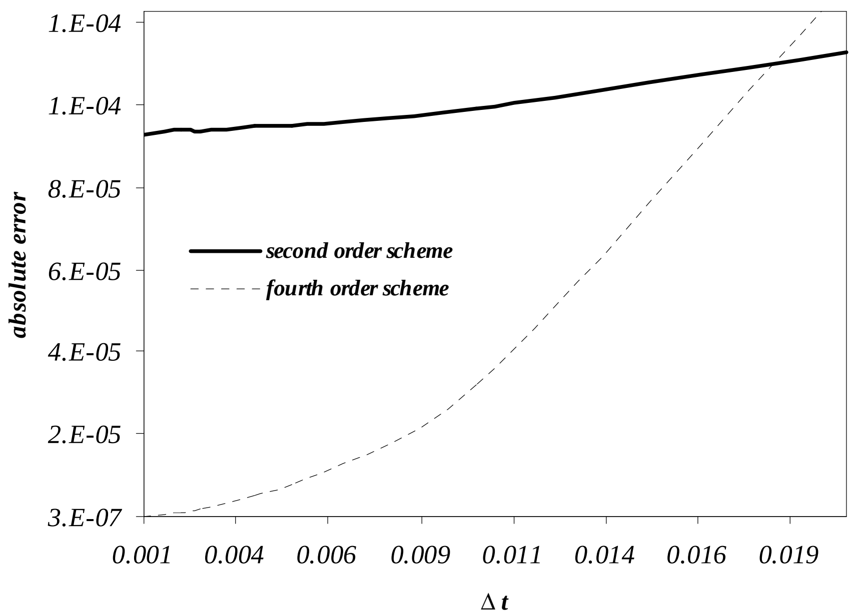

Fig. 2: Absolute error comparison of second and fourth order schemes at $T_q = 1, \quad \alpha = 0.5, \quad S = 0.0$ **,** $\Delta x = \Delta y = 0.05$ **and t =1.0.**

Table.1: The absolute error computed by second order and fourth order scheme with $\Delta x = \Delta y = 0.05$

		Δt=0.002		Δt=0.001	
x	**y**	**second order**	**fourth order**	**second order**	**fourth order**
0.05	0.05	6.20E-06	4.43E-07	6.10E-06	1.11E-07
0.1	0.1	1.99E-05	5.10E-07	1.96E-05	1.27E-07
0.15	0.15	3.82E-05	5.89E-07	3.76E-05	1.45E-07
0.2	0.2	5.93E-05	6.76E-07	5.83E-05	1.65E-07
0.25	0.25	8.17E-05	7.70E-07	8.03E-05	1.86E-07
0.3	0.3	1.04E-04	8.69E-07	1.02E-04	2.07E-07
0.35	0.35	1.25E-04	9.71E-07	1.23E-04	2.28E-07
0.4	0.4	1.45E-04	1.08E-06	1.42E-04	2.48E-07
0.45	0.45	1.61E-04	1.18E-06	1.58E-04	2.67E-07
0.5	0.5	1.73E-04	1.29E-06	1.70E-04	2.85E-07
0.55	0.55	1.80E-04	1.40E-06	1.77E-04	3.03E-07
0.6	0.6	1.81E-04	1.51E-06	1.78E-04	3.19E-07
0.65	0.65	1.77E-04	1.63E-06	1.74E-04	3.35E-07
0.7	0.7	1.65E-04	1.74E-06	1.63E-04	3.50E-07
0.75	0.75	1.47E-04	1.86E-06	1.45E-04	3.64E-07
0.8	0.8	1.22E-04	1.98E-06	1.21E-04	3.78E-07
0.85	0.85	9.06E-05	2.11E-06	8.99E-05	3.93E-07
0.9	0.9	5.54E-05	2.25E-06	5.51E-05	4.10E-07
0.95	0.95	2.11E-05	2.42E-06	2.10E-05	4.32E-07

The results of errors for T_{ij}^{n} at $t=1.0$, computed for $\Delta x=\Delta y=0.05$ and two choices $\Delta t=0.002$ and $\Delta t=0.001$ using both the fourth order and second order finite difference schemes are listed in Table 1. Also, we note that the errors of the fourth order scheme are smaller than those of the second order scheme in both cases.

5.Conclusions

We have introduced a fourth order compact finite difference scheme with Crank-Nicolson technique for solving a two dimensional heat transport equation at the microscale. We proved that the scheme is unconditionally stable with respect to the initial values. Our Numerical results shows that the fourth order compact scheme is computationally more efficient and more accurate than the second order scheme [12].

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اسلوب الفروقات المحددة المضغوطة من الرتبة الرابعة لحل معادلة التوصيل الحراري ثنائية البعد

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المستخلص

في هذا البحث قمنا باستخدام اسلوب الفروقات المحددة المضغوطة من الرتبة الرابعة بالنسبة للحيز والثنائية بالنسبة للزمن مع طريقة كرانك نيكولسون لحل معادلة التوصيل الحراري ثنائية البعد بعد ادخال دالة وسيطية عليها. اثبتنا نظرياً ان هذا الاسوب مستقر بالنسبة لشروطه الابتدائية. قمنا بتطبيق هذا الاسلوب على مثال اختباري لقياس الدقة. ووجدنا ان النتائج العددية التي حصلنا عليها كانت عالية الدقة مقارنة بالاساليب التي تناولتها الدراسات السابقة لحل معادلة التوصيل ثنائية البعد[12].