Journal of Kerbala University , Vol. 9 No.2 Scientific . 2011

Certain Forms of β^{**}-Continuous Functions

ا.د. هادي جابر مصطفى
م. هيام خزعل حسن م. رويدة رزاق محسن
كلية الرياضيات وعلوم الحاسوب
جامعة الكوفة--قسم الرياضيات

Abstract:

In this work, we obtain new weak and strong forms of β^{**}-continuous functions Using the concept of gβ-closed set.
We also obtain a characterization of β-$T_{\frac{1}{2}}$ spaces.

المستخلص:

في هذا البحث قدمنا أنماط ضعيفة وقوية من الدوال المستمرة β^{**} باستخدام مفهوم المجموعات المغلقة gβ.
أيضا حصلنا تمييز للفضاءات β - $T_{\frac{1}{2}}$.

1- Introduction:

In this paper , we introduce Weak form of β^{**}-continuous functions called M-β^{**}-continuous functions by using gβ-closed sets obtain some basic properties of such functions also we introduce and study contra-β^{**}-continuous functions.

This notion is a stronger form of M-β^{**}-continuous functions.

Finally we introduce and study perfectly contra-β^{**}-continuous functions which is a strong form of β^{**}-continuous functions.

Throughout this paper (X, τ) and (Y, σ) (or X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated.

2- Basic definitions:

In this section we recall the basic definitions needed in this work.

2-1 Definition:[1]

Let (X, τ) be a topological space , let A $\subseteq$ X then we say that:

i- A is semi-open if A $\subseteq$ cl Int A . The complement of semi-open set is called semi-closed.

ii- A is α -open if A $\subseteq$ Int cl Int A The complement of α -open set is called α -closed.

iii- A is β -open if A $\subseteq$ cl Int cl A The complement of β -open set is called β -closed.

2-2 **Definition:[2]**

i- the intersection of all semi-closed sets containing A is called the semi-closure of A and is denoted by SclA

ii- the intersection of all α-closed sets containing A is called the α-closure of A and is denoted by α clA.

iii- the intersection of all β-closed sets containing A is called the β-closure of A and denoted by β clA.

2-3 **Definition:[1]**

i- the family of all semi-open sets in X is denoted by SO(X).

ii- the family of all α-open sets in X is denoted by α O(X).

iii- the family of all β-open sets in X is denoted by β O(X).

2-4 **Definition:[2]**

A subset F of (X,τ) is said to be :

i- g- closed in (X,τ)[2] if F$\subseteq$O and O is open $\Rightarrow$cl(F) $\subseteq$O.

ii- gα - closed in (X,τ)[2] if F$\subseteq$O and O is α - open $\Rightarrow \alpha$ cl(F)$\subseteq$O.

iv- gs- closed in (X,τ)[2] if F$\subseteq$O and O is open $\Rightarrow$ scl(F) $\subseteq$O.

v- sg-closed in (X,τ), [2] if F$\subseteq$O and O is semi-open $\Rightarrow$scl(F) $\subseteq$O.

vi- gβ-closed in (X,τ) if F$\subseteq$O and O is β-open $\Rightarrow \beta$ cl(F) $\subseteq$O.

A subset W is said to be (g-open, gα -open, gs-open, sg-open, gβ-open) if its complement W^c=X-W is (g-closed, gα - closed, gs- closed, sg- closed, gβ - closed).

2-5 **Definition:[3]**

A function f : (X,τ)$\longrightarrow$ (Y, σ) is called:

i- β^{**}-continuous[3] if for each v$\in \beta$ O(Y, σ)

(that is v is β-open in Y)we have f^{-1} (v)$\in$ β O (X,τ).(I,e. f^{-1}(v) is β-open in X).

ii- β^{**}- closed if for every β- closed set W of (X,τ), f(W) is β- closed in (Y, σ).

iii- β^{**}-open if for every β-open set W of (X,τ), f(W) is β- open in (Y, σ).

iii- contra-β- closed if f(U) is β-open in Y for each closed set U of X.

3- **Main Results:**

Before, we state the main results of this paper , we introduce the following definitions.

3-1 Definition:

A function f: (X,τ)$\longrightarrow$ (Y, σ) is said to be M-β^{**}-continuous if β cl(F) $\subseteq$ f^{-1}(O) whenever O is a β-open subset of (Y, σ), F is a gβ- closed subset of (X,τ), and F $\subseteq$ f^{-1}(O) .

3-2 Definition:

A function f: $(X,\tau) \to (Y, \sigma)$ is said to be M-β- closed if , f(W) $\subseteq \beta$ intA whenever A is a gβ- open subset of (Y, σ). W is a β- closed subset of (X,τ), and f(W) $\subseteq$ A.

3-3 Theorem:

i- f: $(X,\tau) \to (Y, \sigma)$ is M-β^{**}-continuous if f^{-1}(O) is β- closed in (X,τ) for every β-open O in (Y, σ), (that is if f is contra-β^{**}-continuous) [3].

ii- f: $(X,\tau) \to (Y, \sigma)$ is M-β- closed if f(W) is β-open in (Y, σ) for every β- closed subset W of (X,τ) (that is if f is contra-β^{**}- closed).

Proof:

i- Let F$\subseteq f^{-1}$(O), where O is β- open in (Y, σ) and F is a gβ-closed subset of (X,τ). Therefore βcl(F) $\subseteq \beta$cl(f^{-1}(O))= f^{-1}(O).thus f is M-β^{**}-continuous.

ii- Let f(W) $\subseteq$A, where W is a β- closed subset of (X,τ) and A is a gβ- open subset of (Y, σ), therefore βint(f(W)) $\subseteq \beta$int(A) then f(W) $\subseteq \beta$int(A) thus f is M- β- closed.

3-4 Remark:

i- Clearly β^{**}-continuous functions are M-β^{**}-continuous.

ii- β^{**}-closed functions are M- β- closed.

Proof:

i- let f : $(X,\tau) \to (Y, \sigma)$ be β^{**}-continuous, let O be β- open in (Y, σ), hence f^{-1}(O) is also β- open. Now F is gβ- closed so F$\subseteq f^{-1}$(O)$\Rightarrow \beta$cl(F) $\subseteq f^{-1}$(O).

let f : $(X,\tau) \to (Y, \sigma)$ be β^{**}-closed, let W be β- closed in (X,τ), so f(W) is also β-closed. Now A is gβ- open so f(W) $\subseteq$A $\Rightarrow$ f(W) $\subseteq \beta$ int (A) (take the dual of the definition of gβ- closed).

□

3-5 Theorem:

Let f : $(X,\tau) \to (Y, \sigma)$ be a function from a space (X,τ) to space (Y, σ) :

i- let all subsets of (X,τ) be clopen , then f is M-β^{**}-continuous if and only if f is contra-β^{**}-continuous(that is f^{-1}(O)is β- closed in (Y, σ)).

ii- Let all subsets of (Y, σ) be clopen, then f is M-β- closed if and only if f is contra-β^{**}-closed .(that is f(W) is β- open in (Y, σ) for every β- closed subset W of (X,τ).

Proof:

i- Assume that f is M-β^{**}-continuous. Let A be an arbitrary subset of (X,τ) such that $A \subseteq V$, where V is β- open in (X,τ)then by hypothesis $\beta cl(V)= V$ therefore all subsets of (X,τ) are gβ-closed (and hence all are gβ-open). So for any O which is β-open in (Y, σ), $f^{-1}(O)$ is gβ-closed in (X,τ). Since f is M- β^{**}-continuous, $\beta cl(f^{-1}(O)) \subseteq f^{-1}(O)$, therefore $\beta cl(f^{-1}(O)) = f^{-1}(O)$,

I.e. $f^{-1}(O)$ is β- closed in (X,τ), so f is contra-β^{**}-continuous.

ii- Assume that f is M- β- closed as in (i), we obtain that all subsets of $(Y, \sigma)$are gβ-open. Therefore for any β-closed subset W of (X,τ), f(W) is gβ- open in Y.

since f is M- β- closed, $f(W) \subseteq \beta\, int(f(W))$. Hence $f(W)= \beta\, int(f(W))$, i. e, f(W) is β-open. □

3-6 Corollaries:

Let $f : (X,\tau) \longrightarrow (Y, \sigma)$ be a function from a topological space (X,τ) to a topological space (Y, σ) :

i- Let all subsets of (X,τ) be clopen, then f is M- β^{**}-continuous if and only if f is β^{**}-continuous.

ii- Let all subsets of (Y, σ)be clopen, then f is M-β- closed if and only if f is β^{**}-closed.

3-7 Example:

If $f : X \longrightarrow Y$ is β^{**}-continuous, then f need not be contra-β^{**}-continuous, for example:

The identity function on the topological space $(X,\tau)$where $\tau=\{\phi, X, \{a\}, \{a, b\}\}$, $X=\{a, b, c\}$, is β^{**}-continuous but not contra-β^{**}-continuous.

3-8 Definition:

A function $f : (X,\tau) \longrightarrow (Y, \sigma)$ is called perfectly contra-β^{**}-continuous if the inverse of every β-open set in Y is β-clopen in X.

3-9 Remark:

Every perfectly contra-β^{**}-continuous function is contra-β^{**}-continuous and β^{**}-continuous.

3-10 Theorem:

If a function $f : (X,\tau) \longrightarrow (Y, \sigma)$ is β^{**}-continuous function and M-β- closed, then $f^{-1}(A)$ is a gβ-closed whenever A is a gβ-closed subset of (Y, σ).

Proof :

Let A be a gβ-closed subset of (Y, σ) suppose that f^{-1}(A) $\subseteq$O where O is β-open in (X,τ), now $O^c \subseteq f^{-1}(A^c)$, so f($O^c$)$\subseteq f^{-1}(A^c)$, but f is M-$\beta$- closed then f($O^c$)$\subseteq \beta$int($A^c$) =(βcl(A))^c . It follows that :

$O^c \subseteq f^{-1}(\beta \text{cl(A)})^c$ and hence : $f^{-1}(\beta$cl(A)) $\subseteq$O. □

Since f is β^{**}-continuous, $f^{-1}(\beta$cl(A)) is β-closed thus we have:

βcl(f^{-1}(A)) $\subseteq \beta$cl($f^{-1}(\beta$cl(A))= $f^{-1}(\beta$cl(A)) $\subseteq$O. This implies that f^{-1}(A) is gβ-closed in (X,τ).

3-11 Remark:

Let f : (X,τ)$\longrightarrow$ (Y, σ) and g: (Y, σ)$\longrightarrow$ (Z,γ) be two functions such that g$\circ$f: (X,τ)$\longrightarrow$ (Z,γ) then :

i- g o f is contra-β^{**}-continuous if g is β^{**}-continuous and f is contra-β^{**}-continuous.

ii- g o f is contra-β^{**}-continuous if g is contra -β^{**}-continuous and f is β^{**}-continuous.

3-12 Theorem:

If f : (X,τ)$\longrightarrow$ (Y, σ) is M-β^{**}-continuous and is an open, gβ-closed subset of (X,τ), then the restriction f_A = f/A : (A, τ_A)$\longrightarrow$ (Y, σ) is M-β^{**}-continuous.

Proof:

Assume F is a gβ-closed subset relative to A and G is a β-open subset of (Y, σ) for which F$\subseteq$(f_A)$^{-1}$(G).then F$\subseteq f^{-1}$(G) $\cap$A.

On the other hand , F is gβ-closed in X. since f is M-β^{**}-continuous, then βcl(F) $\subseteq f^{-1}$(G).this implies that βcl(F) $\cap$A $\subseteq f^{-1}$(G) $\cap$A, using that fact that βcl(F) $\cap$A =βcl$_A$(F) [3].

We have : βcl$_A$(F) $\subseteq (f_A)^{-1}$(G)

Thus f_A: (A, τ_A)$\longrightarrow$ (Y, σ) is M-β^{**}-continuous. □

4- A Characterization of β-$T_{\frac{1}{2}}$ Spaces

In this section, we give a characterization of the class of β-$T_{\frac{1}{2}}$ spaces.

4-1 Definition :[4]

A space (X,τ) is said to be β-$T_{\frac{1}{2}}$ space, if every gβ-closed set is β-closed.

4-2 Theorem:

Let (X,τ) be a space, then (X,τ) is a β-$T_{\frac{1}{2}}$ space if and only if f : (X,τ)$\rightarrow$ (Y, σ) is M-β^{**}-continuous, for every space (Y, σ) (and every function f: (X,τ)$\rightarrow$ (Y, σ)).

Proof:

$\Rightarrow$)

Let F be a gβ-closed subset of (X,τ) and F $\subseteq f^{-1}$(O) where O is β-open in (Y, σ) since (X,τ) is a β-$T_{\frac{1}{2}}$ space F is β-closed(I.e. F=βcl(F))

Therefore βcl(F) $\subseteq f^{-1}$(O) and hence f is M-β^{**}-continuous.

$\Leftarrow$)

Let W be a gβ-closed subset of (X,τ) and Y be the set X with the topology σ={ϕ, Y, W}

Let f : (X,τ)$\rightarrow$ (Y, σ) be the identity function by assumption f is M-β^{**}-continuous since W is gβ-closed in (X,τ) and β-open in (Y, σ) and W $\subseteq f^{-1}$(W), it follows that βcl(W) $\subseteq f^{-1}$(W) = W. Hence W is β-closed in (X,τ), therefore (X,τ) is a β-$T_{\frac{1}{2}}$ space. □

References

(1) Abed El-Monsef M. E., β-Opens and β-Continuous Functions, Bull. Fac. Sci, Assint Umr. 12(1983),(77-90).

(2) Levine N., Generalized Closed Sets in Topology, Rend. Cire Mat. 19(1970),(89-96).

(3) Mustafa Hadi J., Central- β-Continuous Functions First Kufa conference (2008), Kufa Univ.

(4) Noiri T., Properties of β-Connected Spaces, Acta Math. Hung, 101(2003),(227-236).