On simultaneous approximation by summation-integral type Beta operator

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Abstract

In this paper, we introduced modified summation-integral Beta operator $R_{n,r}(f;x)$ in the space C_α^r of r-times differentiable functions. The definition of these operators is closely connected with considered functional space. we studied simultaneous approximation for a new sequence of linear positive operator $R_{n,r}(f;x)$. First, we establish the basic pointwise convergence theorem and the proceed to discuss the Voronovsicoskaja type asymptotic formula. Finally, we obtain an error estimate in terms of modulus of continuity of the function being approximated.

Key Words: Approximation by Linear Positive Operators; Voronovsicokaja Theorem; Pointwise Estimate; Simultaneous Approximation.

1. Introduction

Beta operators are important and applied widely in probability theory and approximation theory R.Upreti (1985). Let $f \in C_\alpha[0,\infty) = \{f \in C[0,\infty): |f(x)| \le M(1+t)^\alpha, M, \alpha > 0\}$, with the norm:

$$\|f(.)\| = \sup_{x\in(0,\infty)} |f(x)|. \qquad (1.1)$$

we can define classical Beta $B_n(f;x)$, QI Qiu Lan, GUO Sheng and LI Jian Kun (2009), as:

$$B_n(f;x) = \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x);\ x \in [0,\infty), \qquad n \in N.$$

Where $b_{n,k}(x) = \frac{(n+k)!}{n!(k-1)!} x^k (1+x)^{-n-k-1}$.

The generalized summation- integral type operators with Beta basis functions are widely studied V.Gupta, M.Gupta and V.Vasishtha (2003), is defined as:

$$R_n(f;x) = \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\, f(t)dt\, ;\; f \in C_\alpha[0,\infty).$$

At present, L.Rempulska and Z.Walczak (2004), defined the following sequence:

$$S_{n,r}(f;x) = \sum_{k=0}^{\infty} q_{n,k}(x)\sum_{j=0}^{r}\frac{f^{(j)}\left(\frac{k}{n}\right)}{j!}\left(x-\frac{k}{n}\right)^j,\; x \in [0,\infty), r \in N^0.$$

Where $f \in C_\alpha^r[0,\infty) = \{f \in C_\alpha \text{ } having\ deivtatives\ f^{(k)} \in C_\alpha, k = 1,\dots,r\ \}$ and the norm defined by (1.1) $(C_\alpha^r \equiv C_\alpha)$.

Here, we use a similar technique to introduce a summation-integral Beta operator $R_{n,r}(f;x)$ as follows.

$$R_{n,r}(f;x) = \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\sum_{j=0}^{r}\frac{f^{(j)}(t)}{j!}(t-x)^j\, dt, \quad f \in C_\alpha^r[0,\infty) \qquad (1.2)$$

Note, if $r = 0$, then $R_{n,0}(f;x) = R_n\ (f;x)$. It easily verified that the operator (1.2) are linear positive operator, In the present paper, we prove the convergence theorem for this operator. Then, we study asymptotic formula of the Voronovskaja type and an error estimate in terms of modulus of continuity of the function being approximated.

Before we study the operator $R_{n,r}(f;x)$, we need to know some properties of operator $R_n\ (f;x)$

Lemma 1.1

For $x \in [0,\infty)$ P.Maheswari (2006) and V.Gupta and A.Lupas (2006), we get

1) $\sum_{k=0}^{\infty} b_{n,k}(x) = n;$ 2) $\sum_{k=0}^{\infty} k b_{n,k}(x) = n(n+1)x\,;$

3) $\sum_{k=0}^{\infty} k^2 b_{n,k}(x) = n(n+1)x + n(n+1)(n+2)x^2.$

4) $\int_0^{\infty} b_{n,k}(x)t^m\, dt = \frac{(m+k)!\,(n-m-1)!}{k!\,(n-1)!}, t \in (0,\infty)\ and\ m\ \in N^0$

5) Suppose that $\varphi_{n,m}(x) = \sum_{k=0}^{\infty} k^m b_{n,k}(x)$, then

$$\varphi_{n,m+1}(x) = x(x+1)\varphi_{n,m}(x) + n(n+1)\varphi_{n,m}(x) \qquad (1.3)$$

6) $\sum_{k=0}^{\infty} k^m b_{n,k}(x) = \frac{(n+m)!}{(n-1)!}x^m + \frac{(n+m-1)!}{(n-1)!}x^{m-1}$

$$+trems\ in\ lower\ powers\ of\ x; \qquad (1.4)$$

Lemma 1.2

Let the m-th order moment $E_{n,m}(x)$ for the operator $R_n(f;x)$, QI Qiu Lan, GUO Sheng and LI Jian Kun (2009), be defined as:

$$E_{n,m}(x)=R_n((t-x)^m;x)=\frac{1}{n}\sum_{k=0}^{\infty}B_{n,k}(x)\int_0^{\infty}B_{n,k}(t)\,(t-x)^m dt, n>m \; and \; m\in N^0$$

We have:

1) $E_{n,0}(x)=1,$ $E_{n,1}(x)=\frac{1+2x}{n-1}$ and

$$(n-m-1)E_{n,m+1}(x)$$
$$=2mx(1+x)E_{n,m-1}(x)+x(1+x)E'_{n,m}(x)+(m+1)(1+2x)E_{n,m}(x)$$

2) $E_{n,m}(x)$ is a polynomial in x of degree $\left[\frac{m}{2}\right]$;

3) For every $x\in[0,\infty)$, $E_{n,m}(x)=O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$.

4) $\frac{1}{n}\sum_{k=0}^{\infty}b_{n,k}(x)(k-(n+1)x)^{2j}=O(n^j); j\in N^0$

Lemma 1.3: V.Gupta and A.Lupas (2005) [6]**,** There exist the polynomials $Q_{i,j,r}(x)$ independent of *n* and *k* such that

$[x(1+x)]^r D^r\left[b_{n,k}(x)\right]=\sum_{\substack{2i+j\leq r\\ i,j\geq 0\geq}}(n+1)^i[k-(n+1)x]^j Q_{i,j,r}(x)b_{n,k}(x)$. where $D=\frac{d}{dx}$.

Lemma 1.4 For $m\geq 1$, we have:

$$R_n\,(t^m;x)=\frac{1}{n}\sum_{k=0}^{\infty}b_{n,k}(x)\int_0^{\infty}b_{n,k}(t)\,t^m dt$$
$$=\frac{(n-m-1)!}{n!}\left\{\frac{(n+m)!}{(n-1)!}x^m+\frac{m^2(n+m-1)!}{(n-1)!}x^{m-1}\right.$$
$$\left.+\,term\ containing\ lower\ powers\ of\ x\right\}.$$

Proof:

By using Lemma 1.1, we have:

$$R_n\,(t^m;x)=\frac{1}{n}\sum_{k=0}^{\infty}b_{n,k}(x)\int_0^{\infty}b_{n,k}(t)\,t^m dt=\frac{1}{n}\sum_{k=0}^{\infty}b_{n,k}(x)\frac{(m+k)!\,(n-m-1)!}{k!\,(n-1)!}$$

$$R_n\,(t^m;x)=\frac{(n-m-1)!}{n!}\sum_{k=0}^{\infty}b_{n,k}(x)\left(k^m+\frac{m(m+1)}{2}k^{m-1}\right.$$
$$\left.+\,term\ containing\ lower\ powers\ of\ x\right).$$

Using (1.4), we obtain:

$$= \frac{(n-m-1)!}{n!}\left(\frac{(n+m)!}{(n-1)!}x^m + \frac{m(m-1)}{2}\frac{(n+m-1)!}{(n-1)!}x^{m-1}\right.$$

$$\left. + \frac{m(m+1)}{2}\frac{(n-m-1)!}{(n-1)!}x^{m-1} + term\ containing\ lower\ powers\ of\ x\right)$$

$$= \frac{(n-m-1)!}{n!}\left(\frac{(n+m)!}{(n-1)!}x^m + \frac{m^2(n+m-1)!}{(n-1)!}x^{m-1}\right.$$

$$\left. + term\ containing\ lower\ powers\ of\ x\right).$$

2. Auxiliary Results

Theorem 2.1

For $r \in N^0, n \in N\ and\ x \in [0,\infty)$, the following conditions are hold when $n \to \infty$:

1)$R_{n,r}(1;x) = 1;$

2)$R_{n,r}(t;x) = \left(\frac{2(n+1)}{n-1} - 1\right)x + \frac{1}{n-1};$

3)$R_{n,r}(t^2;x) = \left(\frac{4(n+1)(n+2)}{(n-1)(n-2)} - \frac{4(n+1)}{(n-1)} + 1\right)x^2 + \left(\frac{16(n+1)}{(n-1)(n-2)} - \frac{4}{n-1}\right)x + \frac{8}{(n-1)(n-2)}.$

Therefore by using the well-know Korovkin`s Theorem, we get

$R_{n,r}(f;x) \to f(x)\ as\ n \to \infty$.

Proof: the consequence (1), is easily as follows. The proof of (2), is given below:

$$R_{n,r}(t;x) = \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\sum_{j=0}^{r}\frac{(t-x)^j}{j!}D^j\, t\, dt$$

$$= \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\left\{t + (t+x) + \underbrace{0+0+\cdots+0}_{r-1\ times}\right\}dt$$

By using Lemma 1.1, we get

$$R_{n,r}(t;x) = \left(\frac{2(n+1)}{n-1} - 1\right)x + \frac{1}{n-1} \to x\ as\ n \to \infty.$$

The consequence (3) can be proved by using the same technique.

Therefore, the operator $R_{n,r}(f;x) \to f(x)\ as\ n \to \infty.$

Lemma 2.1

The m-th order moment of the operator $R_{n,r}(f;x)$ is defined as:

$$V_{n,m}(x) = \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\sum_{j=0}^{r}\frac{(t-x)^j}{j!}D^j\,(t-x)^m\, dt.$$

Then

$V_{n,0}(x) = 1, \quad V_{n,1}(x) = 2\,\frac{1+2x}{n-1}$ and

$V_{n,m}(x) = 2^m E_{n,m}(x), \quad m \geq 0$ (2.1)

Further, we have the following consequences of $V_{n,m}(x)$:

(1) $V_{n,m}(x)$ is a polynomial in x of degree $\left[\frac{m}{2}\right]$;

(2) For every $x \in [0,\infty)$, $V_{n,m}(x) = O\left(n^{-\left[\frac{m+1}{2}\right]}\right)$.

Proof

It is easy to show that $V_{n,0}(x) = 1 \; and \; V_{n,1}(x) = 2\,\frac{1+2x}{n-1}$.

Next , we prove (2.1). For $x = 0$ it is clearly holds for all $m \geq 1$.

Now, for $x \in (0,\infty)$and $v \geq m$, we have

$$V_{n,m}(x) = \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\sum_{j=0}^{r}\frac{(t-x)^j}{j!}D^j(t-x)^m\,dt.$$

$$= \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\sum_{j=0}^{m}\frac{(t-x)^j}{j!}D^j(t-x)^m\,dt + 0$$

$$= \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\sum_{j=0}^{m}\binom{m}{j}(t-x)^m\,dt.$$

$$= 2^m\frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\,(t-x)^m dt = 2^m E_{n,m}(x),$$

For which (2.1) is immediate. then by using Lemma (1.2) the consequences (1) and (2) are hold.

Lemma 2.2 For $r \geq m$, we get

$$R_{n,r}(t^m;x) = \sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}\frac{(n-(m-j+i)-1)!}{n!}\left(\frac{(n+m-j+i)!}{(n-1)!}x^m + \frac{(m-j+i)^2(n+m-j+i-1)!}{(n-1)!}x^{m-1} + term\ containing\ lower\ powers\ of\ x\right).$$

Proof:

$$R_{n,r}(t^m;x) = \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_0^{\infty} b_{n,k}(t)\sum_{j=0}^{m}\frac{(t-x)^j}{j!}D^j t^m\,dt.$$

$$= \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_{0}^{\infty} b_{n,k}(t)\sum_{j=0}^{m}\frac{m!}{j!\,(m-j)!}t^{m-j}(t-x)^{j}\ dt.$$

$$= \frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)\int_{0}^{\infty} b_{n,k}(t)\sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{\infty}\binom{j}{i}(-x)^{j-i}t^{m-j+i}\ dt.$$

$$= \sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{\infty}\binom{j}{i}(-x)^{j-i}R_n(t^{m-j+i};x)$$

By using Lemma 1.4, we have

$$= \sum_{j=0}^{m}\binom{m}{j}\sum_{i=0}^{\infty}\binom{j}{i}(-1)^{j-i}\ \frac{(n-(m-j+i)-1)!}{n!}\left\{\frac{(n+m-j+i)!}{(n-1)!}x^{m} + \frac{(m-j+i)^2(n+m-j+i-1)!}{(n-1)!}x^{m-1} + term\ containing\ lower\ powers\ of\ x\right\}$$

Theorem 2.2

Suppose that $r \in N, f \in C_{\alpha}^{r}[0,\infty), \alpha > 0$ and $f^{(r)}$ exists at a point $x \in (0,\infty)$, then

$\lim_{n\to\infty} R_{n,r}^{(r)}(f(t);x) = f^{(r)}(x).$ (2.2)

Further, if $f^{(r)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,\infty), \eta > 0$,

Then (2.2) holds uniformly on $[a,b]$.

Proof: By Taylor`s expansion of f, we get

$$f(t) = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}(t-x)^{i} + \varepsilon(t,x)(t-x)^{r}.$$

Where $\varepsilon(t,x) \to 0$ as $t \to x$, hence

$$R_{n,r}^{(r)}(x) = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}R_{n,r}^{(r)}((t-x)^{i};x) + R_{n,r}^{(r)}(\varepsilon(t,x)(t-x)^{r};x) \coloneqq J_1 + J_2.$$

By using Lemma 2.2, if $j < r$ we have $R_{n,r}^{(r)}\left(t^{i};x\right) = 0$. Hence

$$J_1 = \sum_{i=0}^{r}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}R_{n,v}^{(r)}\left(t^{j};x\right) = \frac{f^{(r)}(x)}{r!}R_{n,r}^{(r)}(t^{r};x) = f^{(r)}(x)\ as\ n \to \infty.$$

Next, making use of Lemma 1.3, we have

$$|J_2| \leq \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} \frac{(n+1)^i}{n} \frac{|Q_{i,j,r}(x)|}{x^r(1+x)^r} \sum_{k=0}^{\infty} b_{n,k}(x)|k$$

$$- (n+1)x|^j \int_0^{\infty} b_{n,k}(t) \sum_{j=0}^{r} \frac{(t-x)^j}{j!} D^j |\varepsilon(t,x)(t-x)^r| dt.$$

Since $\varepsilon(t,x) \to 0\ as\ t \to x$, then for given $\varepsilon > 0$, there exists $\delta > 0$ such that $|\varepsilon(t,x)| < \varepsilon$, whenever $0 < |t-x| < \delta$. For $|t-x| \geq \delta$, there exists a constant $M > 0$ such that $|\varepsilon(t,x)(t-$ *x)r≤M(t−x)α ,α>0.*

Hence,

$$\leq \sup_{\substack{2i+j\leq r\\ i,j\geq 0}} \frac{|Q_{i,j,r}(x)|}{x^r(1+x)^r} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} \frac{(n+1)^i}{n} \sum_{k=0}^{\infty} b_{n,k}(x)|k$$

$$- (n+1)x|^j \left(\varepsilon \int_{|t-x|<\delta} b_{n,k}(x) \varepsilon \sum_{j=0}^{r} \frac{(t-x)^j}{j!} D^j\ |t-x|^r dt \right.$$

$$\left. + M \int_{|t-x|\geq\delta} b_{n,k}(x) \sum_{j=0}^{r} \frac{(t-x)^j}{j!} D^j\ (t-x)^{\alpha} dt \right).$$

$$\leq \sup_{\substack{2i+j\leq r\\ i,j\geq 0}} \frac{|Q_{i,j,r}(x)|}{x^r(1+x)^r} \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} \frac{(n+1)^i}{n} \sum_{k=0}^{\infty} b_{n,k}(x)|k - (n+1)x|^j \left(\varepsilon 2^r \int_{|t-x|<\delta} b_{n,k}(x)|t-x|^r dt \right.$$

$$\left. + M 2^{\alpha} \int_{|t-x|\geq\delta} b_{n,k}(x)(t-x)^{\alpha} dt \right)$$

$$:= J_3 + J_4.$$

Let $\left(\varepsilon 2^r \sup_{\substack{2i+j\leq r\\ i,j\geq 0}} \frac{|Q_{i,j,r}(x)|}{x^r(1+x)^r} \right) = C$ be fixed. Now, using Cauchy-Schwarz inequality for integration and then for summation, we are led to:

$$J_3 \leq C \sum_{\substack{2i+j\leq r\\ i,j\geq 0}} (n$$

$$+ 1)^i \left(\frac{1}{n} \sum_{k=0}^{\infty} b_{n,k}(x)(k-(n+1)x)^{2j} \right)^{1/2} \left(\frac{1}{n} \sum_{k=0}^{\infty} b_{n,k}(x) \int_{|t-x|<\delta} b_{n,k}(t)(t \right.$$

$$\left. - x)^{2r} dt \right)^{1/2} \left(\int_{|t-x|<\delta} b_{n,k}(t)\, dt \right)^{1/2}.$$

From Lemma 1.1 and Lemma 1.2 , we get

$$= C \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} (n+1)^i \, O\left(n^{j/2}\right) O\left(n^{-r/2}\right) = O\left(n^{(r-r)/2}\right) = \varepsilon O(1).$$

Next, Let $\left(M 2^{\alpha} \sup_{\substack{2i+j\leq r \\ i,j\geq 0}} \frac{|Q_{i,j,r}(x)|}{x^r(1+x)^r}\right) = G$ again using Cauchy-Schwarz inequality for integration and then for summation, we get

$$J_4 \leq G \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} (n+1)^i \left(\frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)(k-(n+1)x)^{2j}\right)^{1/2} \left(\frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x) \int_{|t-x|\geq\delta} b_{n,k}(t)(t-x)^{2\alpha} dt\right)^{1/2} \left(\int_{|t-x|\geq\delta} b_{n,k}(t)\, dt\right)^{1/2}.$$

From Lemma 1.1 and Lemma 1.2 , we have:

$$= G \sum_{\substack{2i+j\leq r \\ i,j\geq 0}} (n+1)^i \, O\left(n^{j/2}\right) O\left(n^{-s/2}\right) = O\left(n^{(r-s)/2}\right) = o(1); for\ all\ s > 0, s > r.$$

Now, since $\varepsilon > 0$ arbitrary, it follows that $J_3 \to 0\ as\ n \to \infty$. Also, $J_4 = o(1)$, and hence $J_2 = o(1)$, combining the estimates of $J_1\ and\ J_2$ (2.2) is immediate.

Theorem 2.3

Let $f(t) \in C_{\alpha}^{r}[0,\infty), \alpha > 0$ and $f^{(r+2)}(x)$ exists at a point $x \in (0,\infty)$, then

$$\lim_{n\to\infty} n\left[R_{n,r}^{(r)}(f(t);x) - f^{(r)}(x)\right] = f^{(r)}(x) \sum_{j=0}^{r} \binom{r}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i}(r-j+i)(r-j+i+1)$$
$$+ f^{(r+1)}(x) \sum_{j=0}^{r} \binom{r}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} \left\{2(r-j+i)x + \frac{(r-j+i+1)^2}{r+1}\right\}$$
$$+ f^{(r+2)}(x) \sum_{j=0}^{r} \binom{r}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} \left[x^2 + \left\{\frac{(r-j+i+2)^2}{r+2} - \frac{(r-j+i+1)^2}{r+1}\right\} x\right]$$

Proof. Using Taylor`s expansion of f, we have

$$f(t) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!}(t-x)^i + \varepsilon(t,x)(t-x)^{r+2},$$

Where $\varepsilon(t,x) \to 0\ as\ t \to x$. Then

$$R_{n,r}^{(r)}(f(t);x) = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} R_{n,r}^{(r)}\left((t-x)^i;x\right) + R_{n,r}^{(r)}(\varepsilon(t,x)(t-x)^{r+2};x) = J_1 + J_2.$$

Using the same technique of theorem 2.2, we get $J_2 \to 0$ $as\ n \to \infty$.

$$J_1 = \sum_{i=0}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} R_{n,r}^{(r)}\left(t^j;x\right) = \sum_{i=r}^{r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=r}^{i} \binom{i}{j} (-x)^{i-j} R_{n,r}^{(r)}\left(t^j;x\right);$$

Since $R_{n,v}^{(r)}\left(t^j;x\right) = 0$ $when\ j < r$. Then by using Lemma 2.2, we get

$$J_{1=} \frac{f^{(r)}(x)}{r!} \sum_{j=0}^{r} \binom{r}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} \left\{ \frac{(n-(r-j+i)-1)!}{n!} \frac{(n+r-j+i)!}{(n-1)!} r! \right\}$$

$$+ \frac{f^{(r+1)}(x)}{(r+1)!} \sum_{j=0}^{r} \binom{r}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} \left[\left\{ \frac{(n-(r-j+i)-1)!}{n!} \frac{(n+r-j+i)!}{(n-1)!} r!\,(r+1)(-x) \right\} \right.$$

$$+ \frac{(n-(r-j+i)-2)!}{n!} \left\{ \frac{(n+r-j+i)!}{(n-1)!} (r+1)!\,x \right.$$

$$\left. \left. + \frac{(r-j+i+1)^2 (n+r-j+i+1)!}{(n-1)!} r! \right\} \right]$$

$$+ \frac{f^{(r+2)}(x)}{(r+2)!} \sum_{j=0}^{r} \binom{r}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} \left[\frac{(r+1)(r+2)}{2} \frac{(n-(r-j+i)-1)!}{n!} \frac{(n+r-j+i)!}{(n-1)!} r!\,x^2 \right.$$

$$+ (r+2) \frac{(n-(r-j+i)-2)!}{n!} (-x) \left\{ \frac{(n+r-j+i+1)!}{(n-1)!} (r+1)!\,x \right.$$

$$\left. + \frac{(r-j+i+1)^2 (n+r-j+i+1)!}{(n-1)!} r! \right\}$$

$$+ \frac{(n-(r-j+i)-3)!}{n!} \left\{ \frac{(r+2)!}{2} \frac{(n+r-j+i+2)}{(n-1)!} x^2 \right.$$

$$\left. \left. + \frac{(r-j+i+1)^2 (n+r-j+i+1)!}{(n-1)!} (r+1)!\,x \right\} \right]$$

$$\lim_{n\to\infty} n\left[R_{n,r}^{(r)}(f(t);x) - f^{(r)}(x)\right] = f^{(r)}(x) \sum_{j=0}^{r} \binom{r}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} (r-j+i)(r-j+i+1)$$

$$+ f^{(r+1)}(x) \sum_{j=0}^{r} \binom{r}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} \left\{ 2(r-j+i)x + \frac{(r-j+i+1)^2}{r+1} \right\}$$

$$+f^{(r+2)}(x)\sum_{j=0}^{r}\binom{r}{j}\sum_{i=0}^{j}\binom{j}{i}(-1)^{j-i}\left[x^2+\left\{\frac{(r-j+i+2)^2}{r+2}-\frac{(r-j+i+1)^2}{r+1}\right\}x\right].$$

Theorem 2.4:

Let $f\in C_\alpha^r[0,\infty)$ for some $\alpha>0$ and $r\leq q\leq(r+2)$. if $f^{(q)}(x)$ exists and is continuous on $(a-\eta,b+\eta)$ then for sufficiently large n:

$$\left\|R_{n,r}^{(r)}(f(t);x)-f^{(r)}(x)\right\|_{C[a,b]}$$

$$\leq A_1n^{-1}\sum_{i=r}^{v}\left\|f^{(i)}\right\|_{C[a,b]}+A_2n^{-\frac{1}{2}}\omega_{f^{(v)}}\left(n^{-\frac{1}{2}};(a-\eta,b+\eta)\right)+O(n^{-2}).$$

Where A_1,A_2 are constants independent of f and $n.$ $\omega_f(\delta)$ is the modulus of continuity of f on $(a-\eta,b+\eta)$, and $\|.\|$ denotes the sup-norm on the interval $[a,b]$.

Proof:

By Taylor`s expansion of f, we have

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\xi)-f^{(q)}(\xi)}{v!}(t-x)^q\chi(t)+h(t,x)\big(1-\chi(t)\big).$$

Where ξ lies between t,x and $\chi(t)$ is the characteristic function of the interval $(a-\eta,b+\eta)$.

For $t\in(a-\eta,b+\eta)$ and $x\in[a,b]$, we get:

$$f(t)=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i+\frac{f^{(q)}(\xi)-f^{(q)}(\xi)}{q!}(t-x)^q.$$

For $t\in[0,\infty)\setminus(a-\eta,b+\eta)$ and $x\in[a,b]$, we define

$$h(t,x)=f(t)-\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}(t-x)^i.$$

Now,

$$R_{n,r}^{(r)}(f(t);x)-f^{(r)}(x)$$

$$=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{q!}R_{n,r}^{(r)}\big((t-x)^i;x\big)-f^{(r)}(x)$$

$$+R_{n,r}^{(r)}\left(\frac{f^{(q)}(\xi)-f^{(q)}(\xi)}{q!}(t-x)^q\chi(t);x\right)+R_{n,r}^{(r)}\big(h(t,x)\big(1-\chi(t)\big);x\big)$$

$$:=\Sigma_1+\Sigma_2+\Sigma_3.$$

$$\Sigma_1=\sum_{i=0}^{q}\frac{f^{(i)}(x)}{i!}\sum_{j=0}^{i}\binom{i}{j}(-x)^{i-j}R_{n,r}^{(r)}\big(t^j;x\big)-f^{(r)}(x)$$

By using Lemma 2.2, we get:

$$= \sum_{i=0}^{q} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-x)^{i-j} \frac{d^r}{dx^r} \left[\sum_{j=0}^{m} \binom{m}{j} \sum_{i=0}^{j} \binom{j}{i} (-1)^{j-i} \frac{(n-(m-j+i)-1)!}{n!} \left(\frac{(n+m-j+i)!}{(n-1)!} x^m \right. \right.$$

$$\left. \left. + \frac{(m-j+i)^2 (n+m-j+i-1)!}{(n-1)!} x^{m-1} + O(n^{-2}) \right) \right] - f^{(r)}(x)$$

Consequently,

$\|\Sigma_1\|_{C[a,b]} = A_1 n^{-1} \left[\sum_{i=r}^{q} \left\| f^{(i)} \right\|_{c[a,b]} \right] + O(n^{-2})$, uniformly on $[a,b]$.

To estimate Σ_2, we proceed as follows:

$$|\Sigma_2| \leq R_{n,r}^{(r)} \left(\frac{\left| f^{(q)}(\xi) - f^{(q)}(x) \right|}{q!} |t-x|^q \chi(t); x \right)$$

$$\leq \frac{\omega_{f^{(q)}}\left(\delta; (a-\eta, b+\eta)\right)}{q!} R_{n,r}^{(r)} \left(\left(1 + \frac{|t-x|}{\delta} \right) |t-x|^q; x \right)$$

$$\leq \frac{\omega_{f^{(q)}}\left(\delta; (a-\eta, b+\eta)\right)}{q!} \left[\frac{1}{n} \sum_{k=0}^{\infty} \left| b_{n,k}^{(r)}(x) \right| \int_0^{\infty} b_{n,k}(t) \sum_{j=0}^{r} \frac{(t-x)^j}{j!} D^j \left(|t-x|^q \right. \right.$$

$$\left. \left. + \delta^{-1} |t-x|^{q+1} \right) dt \right], \delta > 0$$

For $s = o, 1, 2, \ldots$, we have

$$\frac{1}{n} \sum_{k=0}^{\infty} b_{n,k}(x) |k - x(n+1)|^j \int_0^{\infty} b_{n,k}(t) \sum_{j=0}^{r} \frac{(t-x)^j}{j!} D^j |t-x|^s dt$$

$$= 2^s \frac{1}{n} \sum_{k=0}^{\infty} b_{n,k}(x) |k - x(n+1)|^j \int_0^{\infty} b_{n,k}(t) (t-x)^s dt \qquad (2.3)$$

$$\leq \frac{2^s}{n} \sum_{k=0}^{\infty} b_{n,k}(x) |k - x(n+1)|^j \left(\int_0^{\infty} b_{n,k}(t) dt \right)^{1/2} \left(\int_0^{\infty} b_{n,k}(t) (t-x)^{2s} dt \right)^{1/2}$$

$$\leq 2^s \left(\sum_{k=0}^{\infty} b_{n,k}(x) \big(k - x(n+1)\big)^{2j} \right)^{1/2} \left(\frac{1}{n} \sum_{k=0}^{\infty} b_{n,k}(x) \int_0^{\infty} b_{n,k}(t) (t-x)^{2s} dt \right)^{1/2}.$$

From Lemma 1.2, we get $\Sigma_2 = O\left(n^{\frac{j}{2}}\right) O\left(n^{\frac{-s}{2}}\right) = O\left(n^{\frac{j-s}{2}}\right)$ uniformly on $[a,b]$.

Therefore, by using Lemma 1.3, (2.3), we get

$$2^s \frac{1}{n}\sum_{k=0}^{\infty}\left|b_{n,k}^{(r)}(x)\right|\int_0^{\infty} b_{n,k}(t)|t-x|^s dt$$

$$\leq 2^s \frac{1}{n}\sum_{k=0}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}(n+1)^i|k+(n+1)x|^j \frac{|Q_{i,j,r}(x)|}{x^r}\ b_{n,k}(x)\int_0^{\infty} b_{n,k}(t)|t$$

$$-x|^s dt \qquad (2.4)$$

$$\leq \left(2^s \sup_{\substack{2i+j\leq r\\ i,j\geq 0}}\ \sup_{x\in[a,b]}\frac{|Q_{i,j,r}(x)|}{x^r(1+x)^r}\right)\left(\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}(n\right.$$

$$\left.+1)^i\left(\frac{1}{n}\sum_{k=0}^{\infty} b_{n,k}(x)|k-(n+1)x|^j\int_0^{\infty} b_{n,k}(x)|t-x|^s dt\right)\right)$$

$= M\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}(n+1)^i O\left(n^{\frac{j-s}{2}}\right) = O\left(n^{\frac{r-s}{2}}\right)$, uniformly on $[a,b]$. Since

$\left(2^s \sup_{\substack{2i+j\leq r\\ i,j\geq 0}} \sup_{x\in[a,b]}\frac{|Q_{i,j,r}(x)|}{x^r(1+x)^r}\right) = M(x)$, but fixed.

Choosing $\delta = n^{\frac{-1}{2}}$ and applying (2.4), we are led to

$$\|\Sigma_2\|_{C[a,b]} \leq \frac{\omega_{f^{(q)}}\left(\delta;(a-\eta,b+\eta)\right)}{q!}\left[O\left(n^{\frac{r-q}{2}}\right)+n^{\frac{1}{2}}O\left(n^{\frac{r-q-1}{2}}\right)\right].$$

$$\|\Sigma_2\|_{C[a,b]} \leq A_2 n^{\frac{-(r-q)}{2}}\omega_{f^{(q)}}\left(\delta;(a-\eta,b+\eta)\right).$$

Since $t\in[0,\infty)\setminus(a-\eta,b+\eta)$, we can choose $\delta>0$ in such a way that $|t-x|\geq\delta$ for all $x\in[a,b]$.

$$|\Sigma_3| \leq 2^s \frac{1}{n}\sum_{k=0}^{\infty}\sum_{\substack{2i+j\leq r\\ i,j\geq 0}}(n+1)^i|k+(n+1)x|^j \frac{|Q_{i,j,r}(x)|}{x^r} b_{n,k}(x)\int_{|t-x|\geq\delta} b_{n,k}(t)|h(t,x)|dt$$

For $|t-x|\leq\delta$, we can find a constant $M>0$ such that $|h(t,x)|\leq M(t-x)^{\alpha}$.

Finally using Schwarz inequality for integration and then for summation, we get $|\Sigma_3| = O(n^{-s}), s>0$ uniformly on $[a,b]$.

Combining the estimates of $\Sigma_1,\Sigma_2,\Sigma_3$ the required result is immediate.

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التقريب المتعدد باستخدام المجموع – تكامل من نوع بيتا

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الخلاصة

في هذا البحث، سوف نقدم تحسين جديد من نوع مجموع – تكامل لمؤثر بيتا والذي سوف نرمز له بالرمز $R_{n,r}(f,x)$ في الفضاء C_α^r القابل للتفاضل الدالي ل r من المرات . وان تعريف هذا المؤثر يكون مغلق ومتصل بالنسبة لفضاء الدوال.

سوف ندرس نظرية التقريب المتعدد بالنسبة للمتتابعة الجديدة للمؤثر الخطي الموجب $R_{n,r}(f,x)$. في البداية سوف نستعرض التقريب النقطي و من ثم نناقش صيغة فورونوفسكي للتقارب لهذا المؤثر.

وأخيرا حصلنا على الخطأ المخمن باستخدام مقياس الاستمرارية للدالة المقربة.

مفتاح الكلمات

التقريب باستخدام المؤثر الخطي الموجب ، نظرية فورونوفسكي ، التخمين النقطي ، التقريب المتعدد.