Generalization of (k, n;f)−arcs of Type (1, n) in PG(2, q)

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Abstract

The main purpose of this paper is to generalize the paper entitled "(k, n; f)- arcs of type (1, n) in PG(2, q)". We generalized the conditions and results of the existence of a (k, n; f)− arcs of type (1, n) in PG(2, q) and we used this generalization to construct a (k, n; f) − arcs of type (m, n) in PG(2, q) . In particular a (9, 6; f)- arc of type (2, 6), when the points of weight 0 form (12, 4)-arc and a (7, 6; f)-arc of type (2, 6), when the points of weight 0 form (14, 4)-arc in PG(2, 4). Finally, we proved there is no (k, n; f) − arcs of type (2,4) in PG(2,4) for points of maximum weights are 3 and 4.

Introduction

The notion of sets weight (k, n)−arcs of projective space of dimension t > 0 was introduced by Scafati (1973), then Barnabe (1979) studied arcs with weighted points . Latter, D'Agostini (1980) studied (k, n; f) − caps of type (n-2, n). Reguso and Rella (1983) studied (k, n; f) −arc of type (1,n) in PG(2, q). Wilson (1986) proved the existence of (10, 7; f) −arc of type (4, 7) in PG(2, 3) and (88, 14; f) −arc of type (11, 14) in PG(2, 9). Hameed (1989) discussed the existence (81, 12; f) −arc of type (9, 12) and (85, 13; f) −arc of type (10, 13) in PG(2, 9). D'Agostini (1994) considered the weighted arcs with nonzero characters. Mahmood(1990), Hameed, Hussen and Abass (2011) counted all the minimal (k, n; f) −arc of type (n-5, n) in PG(2, 5). Abass(2011), examined existence and non–existence of (k, n; f) − arcs of type $(n-3, n)$in PG(2, 9). Finally, Hameed (2012) studied a monoidal (k, q+m; f) –arc of type (m, q+m) in PG(2,q).

Definition 1.1 [Falih]. Let GF(p) = Z/pZ, p prime and let $f(x)$ be irreducible polynomial of degree h over GF(p), then

$$\begin{aligned}\mathrm{GF}(p^h) &= \mathrm{GF}(p)[x]/(f(x)) \\ &= \{ a_0 + a_1 t + \cdots + a_{h-1} t^{h-1} : a_i \text{ in } \mathrm{GF}(p), f(t) = 0\}.\end{aligned}$$

For example, GF(2^2) =$Z_2[x]/(f(x))$ $= \{a + bt: a, b\ in\ Z_2\}$
= { $0,1, t, t^2$: $1 + t + t^2$= $2 = 0$}.
Let $f(x) = x^3+x^2+x + \omega$ be a cyclic irreducible polynomial over GF(4).

Definition 1.2[Hirschfeld]. A projective plane over GF(q) is2-dimentional projective space and denoted by PG(2, q) which contains $q^2 + q + 1$ points, $q^2 + q + 1$ lines, every line contains $q + 1$ points and through every point there pass $q + 1$ lines and satisfy the following axioms:

(i) Any two distinct points determine a unique line ;

(ii) Any two distinct lines intersect in exactly one point ;

(iii) There exist four distinct points such that no three of them are on a line

Definition 1.3 [Hirschfeld]. A(k, n)-arc K in a finite projective plane is a set of k points such that no $(n + 1)$ of them are collinear. A $(k, 2)$−arc denoted by k −arc which is a set of k points such that no three points are collinear. Let ℓ be any line in PG (2, q), if ℓ intersect a (k, n) −arc K in i- points. Then ℓ is called i-secant to K, and let τ_i denoted the total number of i −secants to K in PG$(2, q)$.

Proposition 1. 4 [Hirschfeld]. For a (k, n) −arc K, the following equations are hold:

$$\sum_{i=0}^{n} \tau_i = q^2 + q + 1 \quad (1)$$

$$\sum_{i=1}^{n} i\tau_i = k(q + 1) \quad (2)$$

$$\sum_{i=2}^{n} \frac{i(i-1)}{2} \tau_i = \frac{k(k-1)}{2} \quad (3)$$

Proposition 1.5 [Hirschfeld]. The number of the points of (k, n) −arc K is satisfies the inequality

$$k \leq (n - 1)q + n.$$

Proposition 1.6 [Hirschfeld]. The following statements are true:

(i) If $n|q, q$ is even and $k = (n - 1)q + n$, then k is maximal arc;

(ii) If $n \nmid q$, and $k \leq (n - 1)q + n - 2$, then k is maximum arc .

2. $(k, n; f)$– arcs of type $(1, n)$ of a finite projective plane

Let π be a projective plane of order q and denoted by P, R are respectively the sets of points and lines of π. Let f be a function from P into the set N of non-negative integers and call the weight of $p \in P$ the value $f(p)$ and the support of f the set of points of the plane having non-zero weight. Using f we can define the function $F: R \to Z^+$ such that for any $r \in R$,

$$F(r) = \sum_{p \in r} f(p)$$

We call $F(r)$ the weight of line r.

Definition 2.1[Raguso and.Rella]. A $(k, n; f)$ −arc of the plane π is a subset K of the points of the plane such that:

(i) K is the support of f;

(ii) $k = |K|$;

(iii) $n = \max\{F(r): r \in R\}$. Observe that a (k, n) −arc K is a $(k, n; f)$ −arc if f is chosen as follows:

$$f(p) = \begin{cases} 0, & if\ p \notin K \\ 1, & if\ p \in K \end{cases}$$

Definition 2.2 [Hameed, 1989]. The characters of a $(k, n; f)$ −arc are the integers t_i $(|t_i| \neq 0)$ which denote to the number of lines of weight i for $i = 0, 1, \ldots, n$.

Definition 2. 3 [Hameed**,** 1989]. Denote:

(i) $l_j = |f^{-1}(j)|$ the number of points having weight j for $j = 0, 1, \ldots, \omega$, where $\omega = max_{p \in P} f(p)$;

(ii) V_i^j is the number of lines of weight i through a point of weight j;

(iii) $W = \sum_{j=1}^{\omega} j\, l_j = \sum_{p \in P} f(p)$; by counting the total weight of K;

(iv) $k = \sum_{j=1}^{\omega} l_j$, by counting the points of weight j.

Definition 2. 4 [Hirschfeld]. A$(k, n; f) -$ arc is called monoidal if Im$f = \{0, 1, \omega\}$ and $l_\omega = 1$.

We summarizes the properties of a $(k, n; f) -$ arc of type $(1, n)$ by the following Propositions[Hirschfeld]:

Proposition 2. 5. We have Im$f \leq$ {0, 1, ω} , where $\omega \geq 2$.

Proposition 2. 6. If p is any point of the plane, then $\sum_{r \in [p]} F(r) = W + qf(p)$, where $[p]$ denotes the set of lines through .

Proposition 2. 7. The weight W of $(k, n; f)$ −arc K of type $(1, n)$ satisfies:

$$q + 1 \leq W \leq (n - \omega)(q + 1) + \omega = (n - \omega)q + n$$

Proposition 2. 8. Let K be a $(k, n; f)$ −arc of type (1,n), $m > 0$ and let p a point having weigh s, then $V_1^s\ and\ V_n^s$ are determined independently of p and given by:

(i) $V_1^s = \frac{q(n-s) - W + n}{n-1}$;

(ii) $V_n^s = \frac{q(s-1) + W - 1}{n-1}$.

Proposition 2.9. The necessary condition for existence a $(k, n; f)$ −arc K of type $(1, n)$ is that:

$$q \equiv 0\ mod\ (n - 1)$$

Proposition 2.10. The characters of a $(k,n;f)$ −arc K of type $(1,n)$ are given by:

(i) $t_1 = \frac{q+1}{n-1}(n\frac{q^2+q+1}{q+1} - W)$;

(ii) $t_n = \frac{q+1}{n-1}(W - \frac{q^2+q+1}{q+1})$.

Proposition 2. 11. The weight W of the plane must be a root of the following equation of degree two:

$$W^2 - W[n(q+1)+1] + n(q^2+q+1) + q\omega(\omega-1)l_\omega = 0. \quad (4)$$

Proposition 2. 12. (i) The necessary condition for the existence a $(k,n;f)$ −arc of type (1, n) in projective plane π of order q with $\omega \geq 2$ is that the total weight of π is:

$$W = (n-\omega)q + n \,.$$

(ii) By (i) it is necessary to assume that $W = (n-\omega)q + n$ with $2 \leq \omega \leq n-1$, in this case we have:

$$V_1^0 = \frac{q\omega}{n-1}, \qquad V_1^1 = \frac{q(\omega-1)}{n-1}, \qquad V_1^\omega = 0;$$

$$V_n^0 = q - \frac{q\omega}{n-1} + 1, \quad V_n^1 = \frac{q(n-\omega)}{n-1} + 1, \quad V_n^\omega = q+1.$$

(iii) To calculate parameters l_1, l_ω observe that, because the total weight W of the plane is a solution of the equation (4), we obtained:

$$((n-\omega)q+n)^2 - [(n-\omega)q+n][n(q+1)+1] + n(q^2+q+1) + q(\omega-1)\, l_\omega = 0$$

Thus,

$$l_\omega = (q(n\omega - \omega^2 - n) + \omega(n-1))/\omega(\omega-1)$$

And by using the equation $W = \sum_{i=1}^{\omega} i l_i$, it follows that $l_1 = (q\omega - n + 1/\omega - 1) + 1$.

(iv) The necessary condition for the existence a $(k,n;f)$ −arc of type $(1,n)$ in projective plane π of order q with $\omega \leq 2$ is ω/nq and $(\omega-1)/(q\omega - n + 1)$.

3. Properties of $(k,n;f)$ −arc of type (m,n)

Lemma 3.1. [Hameed, 1989].

(i) Im$f \leq \{0, m, \omega\}$, where $2 \leq \omega \leq n-m$;

(ii) If p is any point of the plane, then $\sum_{r\in[p]} F(r) = W + qf(p)$, where $[p]$ denote the set of lines through p;

(iii) The weight W of a $(k, n; f)$ −arc satisfies $m(q + 1) \leq W \leq (n - \omega)q + n$;

(iv) $V_m^s = (q(n - s) - W + n)/(n - m)$, and $V_n^s = (q(s - m) + W - m)/(n - m)$;

(v) $q \equiv 0 \; mod \; (n - m)$;

(vi) $t_m = (q + 1/n - m)[n (q^2 + q + 1/q + 1) - W]$, and

$t_n = (q + 1/n - m)[W - m (q^2 + q + 1 / q + 1)]$;

(vii) The weight W of the plane π must be a root of the following equation of degree two:

$$W - W[n(q + 1) + m] + nm(q^2 + q + 1) + q\omega(\omega - m)l_\omega = 0.$$

Lemma 3. 2. The weight W of a $(k, n; f)$ −arc K of type (m, n) satisfies

$$m(q + 1) \leq W \leq (n - \omega)(q + 1) + \omega .$$

Proof . Let p any point such that $f(p) = \omega$. Counting the weights of lines r, then we have:

$$\sum_{r\in[p]} F(r) = W + qf(p) \leq n(q + 1) ;$$

$$W \leq n(q + 1) - q\omega;$$

Thus,

$$W \leq (n - \omega)q + n .$$

If Q is a point and $f(Q) = 0$, counting the weight of the lines through Q , therefore, we get:

$$W \geq m(q + 1) .$$

Theorem 3. 3. Let K be a $(k, n; f)$ −arc of type (m, n), $m > 0$ and let p be a point having weight s, then V_n^s *and* V_m^s are determined of p and are given by:

(i) $V_m^s = \frac{q(n-s)-W+n}{n-m}$;

(ii) $V_n^s = \frac{q(s-m)+W-m}{n-m}$.

Proof . (i) By counting the lines through p we get:

$$V_m^s + V_n^s = q + 1 \quad (5)$$

And by counting the weights of the lines through p we get:

$$mV_m^s + nV_n^s = \sum_{r\in[p]} F(r) = W + qf(p) \quad (6)$$

Let $f(p) = s$ and then subtract (6) from the result of multiplication (5) by n, it follows that:

$$V_m^s = \frac{q(n - s) - W + n}{n - m}$$

(ii) By the same argument, subtract (6) from the multiplication result of (5) by m, we get :

$$V_n^s = \frac{q(s-m) + W - m}{n-m}$$

According to the Theorem 3.2, when $s = \{0, m, \omega\}, 2 \leq \omega \leq n-m$, and $W = (n-\omega)q + n$, we obtain the following:

$$V_m^0 = \frac{\omega q}{n-m}, \quad V_m^m = \frac{q(\omega - m)}{n-m}, \quad V_m^\omega = 0;$$

$$V_n^0 = q + 1 - \frac{q\omega}{n-m}, \quad V_n^m = \frac{q(n-\omega)}{n-m} + 1, \quad V_n^\omega = q + 1 .$$

Corollary 3. 4. The necessary condition for the existence of $(k, n; f)$ −arc K of type (m, n) is that :

$$q \equiv 0 \; mod \; (n-m).$$

Proof . The equation $(n-m)\left(V_n^j - V_n^i\right) = q(j-1)$ is valid for any i, j, in particular, it must hold for the case $i = 0, j = \omega$, so we obtain

$$(n-m)(V_n^\omega - V_n^0) = q\omega;$$

It also must hold for case $i = 1, j = \omega$, consequently we get

$$(n-m)(V_n^\omega - V_n^1) = q\omega - q;$$

Thus, $$(n-m)(V_n^1 - V_n^0) = q.$$

Therefor , $q \equiv 0 \; mod \; (n-m)$.

Lemma 3. 5. The characters of a $(k, n; f)$ −arc of type (m, n) are given by

$$t_m = \frac{q+1}{n-m}\left(n\frac{q^2+q+1}{q-1} - W\right);$$

$$t_n = \frac{q+1}{n-m}\left(W - m\frac{q^2+q+1}{q+1}\right).$$

Proof. We have:

$$t_m + t_n = q^2 + q + 1$$

And,

$$mt_m + nt_n = W(q+1).$$

By using the same technique in Theorem (3. 3) we get:

$$(n-m)t_n = W(q+1) - m(q^2+q+1);$$

$$t_n = \frac{q+1}{n-m}\left(W - m\frac{q^2+q+1}{q+1}\right) \tag{7}$$

Also,

$$(n-m)t_m = n(q^2+q+1) - W(q+1)$$

Hence,

$$t_m = \frac{q+1}{n-m}\left(n\frac{q^2+q+1}{q+1} - W\right) \tag{8}$$

Lemma 3. 6. The total weight W of the plane must be a root of the following equation of degree two :

$$W^2 - W[n(q+1)+m)] + nm(q^2+q+1) + q\omega(\omega-m)l\omega = 0.$$

Proof. From the relation $\sum_{j=2}^{n}\binom{j}{2}t_j = \binom{W}{2} + q\sum_{i=2}^{\omega}\binom{i}{2}l_i$;

For Imf={0, m, ω} and $2 \le \omega \le n-m$, it follows that :

$$\binom{n}{2}t_n + \binom{m}{2}t_m = \binom{W}{2} and\ + q\binom{m}{2}l_m + q\binom{\omega}{2}l_\omega$$

$$n(n-1)t_n + m(m-1)t_m = W(W-1) + qm(m-1)l_m + q\omega(\omega-1)l_\omega \tag{9}$$

Now, making use the relation $\sum_{j=1}^{\omega} jl_j = W$, Lemma (3.5), and equation (9) we can conclude that:

$$W^2 - W[n(q+1)+m] + nm(q^2+q+1) + q\omega(\omega-m)l_\omega = 0 \tag{10}$$

Lemma 3. 7. For a maximal $(k,n;f)$ −arc of type (m,n) in PG(2, q) when Im$f = \{0, m, \omega\}$, $m \le n-2$, and $2 \le \omega \le n-$m, we have,

(i) $l_0 = q^2+q+1-(l_m+l_\omega)$;

(ii) $l_m = W - \omega l_\omega/m$;

(iii) $l_\omega = (nq/\omega) - ((\omega q - n + m)/\omega - m)$.

Proof . To calculate the parameters l_ω, observe that because the total weight W of the plane is a solution of equation (10) so we get:

$[(n-\omega)q+n]^2 - [(n-\omega)q+n][n(q+1)+m] + nm(q^2+q+1) + q\omega(\omega-m)l_\omega = 0$,

where $l_\omega = (nq/\omega) - ((\omega q - n + m)/(\omega - m))$ (11)

According to the Definition 2.3, it follows that

$$l_m = W - \omega l_\omega/m \tag{12}$$

Finally, form the relation $\sum_{j=0}^{\omega} l_j = q^2+q+1$, it follows that

$$l_0 = q^2+q+1-(l_m-l_\omega) \tag{13}$$

Theorem 3.8. From the equation (11) the necessary condition for the existence a maximal $(k,n;f)$ −arc of type (m,n) in PG(2, q), with $2 \le \omega \le n-m$ is that:

(i) ω/nq, and $(\omega-m)/(\omega q - n + m)$;

(ii) $\dfrac{nq}{\omega} > \dfrac{\omega q - n + m}{\omega - m}$.

4. $(k, 6; f)$ −arc of type (2, 6) in PG(2, 4)

4. 1 The case l_0 =12.

Lemma 4. 1. 1 Let K be an $(9, 6; f)$ −arc of type (2, 6), then we have the following:

$$V_2^0 = 3, \quad V_2^2 = 1, \quad V_2^3 = 0 ;$$
$$V_6^0 = 2, \quad V_6^2 = 4, \quad V_6^3 = 5 .$$

Proof . Directly , by using Theorem (3. 3) and putting $\omega = 3, n = 6, q = 4$.

Theorem 4.1.2. For existence of $(9, 6, f)$ −arc of type (2, 6) in PG(2, 4) we have the follo- wing:

(i) The number of 6-weighting lines t_6 is 12;

(ii) The number of 2-weighting lines t_2 is 9 ;

(iii) The number of points of weight 2 l_2 is 9 ;

(iv) The number of points of weight 3 l_3 is 0 .

Proof . Directly, from the equations (7), (8), (12) and (11), we respectively get $t_6 = 12, t_2 = 9, l_2 = 9$ and $l_3 = 0$.

Corollary 4. 1. 3. There are no points of weight 3 on any 6- weighting lines and on any 12- weighting lines.

4. 2 Classification of lines of the plane with respect to the $(9, 6; f)$ −arc of type (2, 6).

Let U_6 be 6-weighting lines having on it α points of weight 0 , β points of weight 2, then by counting points of n-weighting lines, we get:

$$\alpha + \beta = q + 1 = 5$$

And summing the weights of the points on U_6, gives $2\beta = n = 6$, so $\beta = 3$, and $\alpha = 2$.

Now, let U_2 be 2-weighting lines having on it α points of weight 0 and β points of weight 2, then $\alpha + \beta = 5$, and $2\beta = 2$, so $\beta = 1$ and $\alpha = 4$.

We summaries the above results in the following table (4. 2.1):

Table (4. 2. 1)

Type of lines	β	α
U_6	3	2
U_2	1	4

Theorem 4. 2. 1. The lines of PG(2, 4) are partitioned into two sets with respect to the maximal $(9, 6; f)$ −arc of type (2, 6) as follows:

(i) U_6^2 which contains 3 points of weight 2, 2 points of weight 0 and no points of weight 3;

(ii) U_2^4 which contains 1 point of weight 2, 4 points of weight 0 and no points of weight 3 .

Proof. From Corollary (4. 1. 3) and Table (4. 2. 1) .

4. 3 The case of maximal $(9,6;f)$ −arc when the points of weight 2 form a (9, 3)- arc .

We discuss the existences and non- existence of $(9,6;f)$ −arc of type (2, 6) when the points of weight 2 for a (9, 3)-arc in PG(2, 4) . Firstly we begin with some results on (9, 3)-arc from proposition (1. 4) we get:

$$\tau_0+\tau_1+\tau_2+\tau_3=21;$$
$$\tau_1+2\tau_2+3\tau_3=45;$$
$$\tau_2+3\tau_3=36\ ,$$

where τ_i; i=0, 1, 2, 3 the number of i-secant of (9, 3)-arc in PG(2, 4).

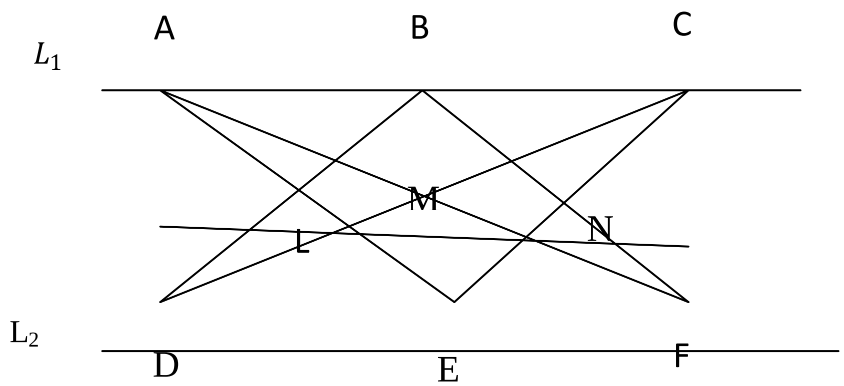
Theorerm 4. 3. 1. The number of τ_0 and τ_2 for (9, 3)-arc formed by the points of weight 2 is zero .

Proof. From the table (4. 2. 1), the 6-weighting lines of the $(9,6;f)$ −arc are 3-secant of (9, 3)-arc and 2-weighting lines of $(9,6;f)-arc$ are 1-secant of (9, 3)-arc , so $\tau_0=\tau_2=0$. Consequently, from Lemma (4. 1. 2) we get $\tau_1=t_2=9$ and $\tau_3=t_6=12$.

Theorem 4. 3. 2. From the proposition (1. 6), a (9, 3)-arc is a maximum arc in PG(2, 4) .

Recall the following well known theorem:

Theorem (Pappus). Given two lines L_1, L_2 in PG(2, 4) such that A, B, C on L_1, and D, E, F on L_2, then N=BF∩CE, M=AF∩CD, and L=AE∩BD, are collinear see Fig(4. 3. 1),



Fig(4. 3. 1)

where the points {1, 2, 4, 5, 8, 11, 12, 13, 14} in PG(2, 4) are Pappus configuration**,** such that A=(1,0,0), B=(0,1,0), C=(1, ω,0), D=(1,ω,1), E=(1,0, ω), F=(1,ω^2,ω^2).

Theorem 4. 3. 3. There is $(9, 6; f)$ $-$arc of type (2, 6) in PG(2, 4) where points of weight two form a (9, 3)-arc of type (12, 0, 9, 0) and points of weight zero form a (12, 4)-arc of type (9, 0, 12, 0, 0).

4. 4 The case $l_0 = 14$

Lemma 4. 4. 1. Let K be a $(7, 6; f)$ $-$arc of type (2, 6), then we have the following:

(i) $V_2^0 = 4,$ $V_2^2 = 2,$ $V_2^4 = 0;$

(ii) $V_6^0 = 1,$ $V_6^2 = 3,$ $V_6^4 = 5;$

(iii) $t_6 = 7,$ $t_2 = 14;$

(iv) $l_2 = 7,$ $l_4 = 0$.

Proof. Put n=6 , q=4, and ω=4 and then using Theorem (3. 3), we get (i) and (ii) , from Lemma (3.5) we obtain (iii) and finally, from Lemma (3. 7) we obtain (iv) .

4. 5. The case of a maximal $(7, 6; f)$ $-$arc when the point of weight 2 form a (7, 3)-arc in PG(2, 4).

From the table (4. 2. 1) and Lemma (4. 4. 1) we get $\tau_0 = \tau_2 = 0, \tau_3 = t_6 = 7$, and $\tau_1 = t_2 = 14$.

Theorem 4. 5. 1. A (7, 3)-arc represents subplane from the plane of order 4.

Theorem 4.5.2. There is $(7, 6; f)$ $-$arc of type (2, 6) in PG(2, 4) when points of weight two form a (7, 3)-arc of type (7, 0, 14, 0) and points of weight zero form (14, 4)-arc of type (14, 0, 7, 0, 0).

Theorem 4.5.3. There is no $(k, n; f)$ $-$arc of type (2, 4) in PG(2, 4) for ω=2.

Proof . Directly, from Theorem (3. 8) by putting $n = 4, q = 4$ and $\omega = 2$.

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الخلاصة :

الهدف الرئيسي من هذا البحث هو تعميم البحث المعنون بالاقواس الموزونة (k,n) من النوع $(1,n)$ في المستوي الاسقاطي من الرتبة q. فقد عممنا الشروط والنتائج لوجود الاقواس الموزونة (k,n)من النوع $(1,n)$ في$PG(2,q)$ واستخدمنا التعميم لبناء اقواس موزونة (k,n)من النوع (m,n) في المستوي الاسقاطي من الرتبة q . حيث برهنا وجود القوس الموزون $(9,6)$ من النوع *(2, 6)* عندما النقاط وزن صفر تشكل القوس $(12,4)$ ،والقوس الموزون $(7,6)$ من النوع $(2,6)$ عندما النقاط وزن صفر تشكل القوس $(14,4)$ في المستوي الاسقاطي من الرتبة الرابعة . وكدالك اثبتنا عدم وجود اقواس موزونة (k,n) من النوع $(2,4)$ في المستوي الاسقاطي من الرتبة الرابعة عندما يكون أكبر وزن للنقاط 3أو 4 .