
Approximation by q -Szász-Mirakjan Operators

Ali J. Mohammad¹ and Hussein Kh. Ghadhban²,

Department of Mathematics, College of Education, University of Basrah, Basrah, IRAQ.

Email¹: alijasmoh@yahoo.com

Email²: hussein.ash93@yahoo.com

Abstract

In this paper, we introduce and study a new type of q -Szász Mirakjan operators $W_n(f, q_n; x)$ in the case $q_n \in (0, 1)$. First, we prove that these operators are converging to the function being approximated. Then, we define the m -th order of q -moments and find a recurrence relation for these q -moments. Finally, we prove Voronovskaja-type asymptotic formulas for the operators $W_n(f, q_n; x)$.

Keywords: q -integers, q -Szász-Mirakjan operators, positive linear operators, Voronovskaja-type asymptotic formula.

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Introduction:

The approximation of functions by using linear positive operators introduced via q -Calculus is currently under intensive research. The pioneer work has been made by [Lupas, 1987] and [Phillips, 1997] they proposed and studies generalizations of Bernstein polynomials involving q -integers. After that, [Mahmudove, 2010] defined the m -th order q -moments for the q -Bernstein polynomials. [Aral, 2008] introduced and studied the q -Szász-Mirakjan operators. [Aral and Gupta, 2006] and [Mahmudove, 2010] study a modifications for q -Szász-Mirakjan operators.

This paper is organized as follows: In section 2, we give standard notations that will be used throughout the paper, introduce the q -Szász operators. Section 3, gives some lemmas which are helping us to study the convergence for the operators $W_n(f, q_n; x)$. Section 4, imports the

q -moments of these operators. Section 5, derives the recurrence relation for the q -moments. Finally section 6, gives the quantitative Voronovskaja-type asymptotic formula.

2. Construction the operators $W_n(f, q_n; x)$:

Throughout the paper, we employ the standard notations of q -calculus, for more details see [Ernst, 2000] and [Kac and Cheung, 2000].

Let $q_n \in (0,1)$. For any $n \in N^0 = \{0,1, \dots\}$ the q -integer $[n]$ and the q -factorial $[n]!$ respectively defined by:

$$[n] = 1 + q_n + q_n^2 + \dots + q_n^{n-1} (n = 1,2,3, \dots), \quad [0] = 0;$$

$$[n]! = [1][2][3] \dots [n-1][n] (n = 1,2,3, \dots), \quad [0]! = 1.$$

For given integers n, k the q -binomial or the Gaussian coefficients are defined by:

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]! [n-k]!}, \quad (0 \leq k \leq n) \text{ and } \begin{bmatrix} n \\ k \end{bmatrix} = 0, \text{ otherwise.}$$

The q -derivative of the function $f(x)$, denoted by $D_q f$, is defined as:

$$(D_q f)(x) = \frac{f(qx) - f(x)}{(q-1)x}, \quad x \neq 0, \quad q \in R^+ \setminus \{1\}.$$

Note that, $D_q(u(x) + v(x)) = D_q(u(x)) + D_q(v(x))$.

The formula for the q -derivative of a product of two functions is defined by:

$$D_q(u(x) v(x)) = D_q(u(x)) v(x) + u(qx) D_q(v(x)).$$

The q -analuge of $(t-x)^m$ is defined as:

$$(t-x)_{q_n}^m = (t-x)(t-q_n x)(t-q_n^2 x) \dots (t-q_n^{m-1} x) = \prod_{j=0}^{m-1} (t-q^j x).$$

The q -exponential function is defined as $e_{q_n}(x) = \sum_{k=0}^{\infty} \frac{x^k}{[k]!}$ such that

$$D_{q_n}(e_{q_n}(ax)) = a e_{q_n}(ax) \text{ and } D_{1/q_n}(e_{q_n}(x)) = e_{q_n}(q_n^{-1}x).$$

Let $C[0, \infty)$ denote the class of all continuous functions on the interval $[0, \infty)$. For $\alpha > 0$, and $f \in C_\alpha[0, \infty) = \{f \in C[0, \infty): |f(t)| \leq C e^{\alpha t} \text{ for some } C > 0\}$.

Definition 1:

Let $0 < q_n < 1, x \in [0, \infty), (p \geq 0), n \in N, k \in N^0 = \{0,1, \dots\}, q_n \rightarrow 1$ and $q_n^n \rightarrow C$, as $n \rightarrow \infty$, where C is a constant. For $f \in C_\alpha[0, \infty)$, we define the operator as;

$$W_n(f(t), q_n; x) = \sum_{k=0}^{\infty} Y_{n,k}(q_n; x) f\left(\frac{[k]}{[n]}\right) \quad (2.1)$$

where, $Y_{n,k}(q_n; x) = \frac{1}{q_n^{k(k-1)/2}} \frac{[n+p]^k x^k}{[k]!} e_{q_n}(-[n+p]q_n^{-k}x)$.

3. Some properties of the weight functions $Y_{n,k}(q_n; x)$.

The next lemmas expose some importance properties of the weight function $Y_{n,k}(q; x)$.

Lemma 1: For $x \in [0, \infty)$, ($p \geq 0$), and $m \in N^0$, we have:

$$(i) \sum_{k=0}^{\infty} Y_{n,k}(q_n; x) = 1; \quad (ii) xD_{q_n} Y_{n,k}(q_n; x) = ([k] - [n + p]x)Y_{n,k}(q_n; x); \quad (3.1)$$

$$(iii) W_n(t^{m+1}, q_n; x) = \frac{x}{[n]} D_{q_n} W_n(t^m, q_n; x) + \frac{q_n^n [p]}{[n]} x W_n(t^m, q_n; x) + x W_n(t^m, q_n; x); \quad (3.2)$$

(iv) Suppose that $\varphi_{n,m}(q_n; x) = \sum_{k=0}^{\infty} [k]^m Y_{n,k}(q_n; x)$, then

$$\varphi_{n,m+1}(q_n; x) = xD_{q_n} \varphi_{n,m}(q_n; x) + [n + p]x\varphi_{n,m}(q_n; x); \quad (3.3)$$

$$\text{and } \varphi_{n,m}(q_n; x) = [n + p]^m x^m + \left(\sum_{j=0}^{m-1} (m - 1 - j) q_n^j \right) [n + p]^{m-1} x^{m-1} + T.L.P.(x); \quad (3.4)$$

where; $T.L.P.(x)$ is the terms in lower powers of x .

Proof: (i) For a fixed $x \in [0, \infty)$, the q -Taylor's expansion [Kac and Cheung, 2000], for the function $g(t)$, is

$$g(t) = \sum_{k=0}^{\infty} \frac{(t - x)_{1/q_n}^k}{[k]!_{1/q_n}} D_{1/q_n}^k g(x).$$

Also we have; $(-x)_{1/q_n}^k = (-1)^k x^k q_n^{-k(k-1)/2}$.

And $D_{1/q_n}^k e_{q_n}(-[n + p]x) = (-1)^k [n + p]^k q_n^{-k(k-1)/2} e_{q_n}(-[n + p]q_n^{-k}x)$.

Hence we get,

$$\begin{aligned} 1 = e_{q_n}(0) &= \sum_{k=0}^{\infty} \frac{(-x)_{1/q_n}^k}{[k]!_{1/q_n}} D_{1/q_n}^k e_{q_n}(-[n + p]x) \\ &= \sum_{k=0}^{\infty} \frac{1}{q_n^{k(k-1)/2}} \frac{[n + p]^k x^k}{[k]!} e_{q_n}(-[n + p]q_n^{-k}x) = \sum_{k=0}^{\infty} Y_{n,k}(q_n; x). \end{aligned}$$

(ii) $x D_{q_n} Y_{n,k}(q_n; x)$

$$= \frac{[k]}{q_n^{k(k-1)/2}} \frac{[n+p]^k x^k}{[k]!} e_{q_n}(-[n+p]q_n^{-k}x) - \frac{1}{q_n^{k(k-1)/2}} \frac{[n+p]^k q_n^k x^{k+1}}{[k]!} [n+p]q_n^{-k} e_{q_n}(-[n+p]q_n^{-k}x) = ([k] - [n+p]x) Y_{n,k}(q_n; x).$$

(iii) $x D_{q_n} W_n(t^m, q_n; x) = \sum_{k=0}^{\infty} \left(\frac{[k]}{[n]}\right)^m ([k] - [n+p]x) Y_{n,k}(q_n; x)$

$$= [n] \sum_{k=0}^{\infty} \left(\frac{[k]}{[n]}\right)^m \left(\frac{[k]}{[n]} - x\right) Y_{n,k}(q_n; x) - q_n^n [p]x \sum_{k=0}^{\infty} \left(\frac{[k]}{[n]}\right)^m Y_{n,k}(q_n; x) = [n] \sum_{k=0}^{\infty} \left(\frac{[k]}{[n]}\right)^{m+1} Y_{n,k}(q_n; x) - [n]x \sum_{k=0}^{\infty} \left(\frac{[k]}{[n]}\right)^m Y_{n,k}(q_n; x) - q_n^n [p]x \sum_{k=0}^{\infty} \left(\frac{[k]}{[n]}\right)^m Y_{n,k}(q_n; x) = [n]W_n(t^{m+1}, q_n; x) - q_n^n [p]x W_n(t^m, q_n; x) - [n]x W_n(t^m, q_n; x).$$

(iv) $x D_{q_n} \varphi_{n,m}(q_n; x) = \sum_{k=0}^{\infty} [k]^m ([k] - [n+p]x) Y_{n,k}(q_n; x)$

$$x D_{q_n} \varphi_{n,m}(q_n; x) = \sum_{k=0}^{\infty} [k]^{m+1} Y_{n,k}(q_n; x) - [n+p]x \sum_{k=0}^{\infty} [k]^m Y_{n,k}(q_n; x)$$

$$\varphi_{n,m+1}(q_n; x) = x D_{q_n} \varphi_{n,m}(q_n; x) + [n+p]x \varphi_{n,m}(q_n; x).$$

We prove (3.4), by induction on m , From(3.3), we get:

$$\varphi_{n,1}(q_n; x) = [n+p]x, \quad \varphi_{n,2}(q_n; x) = [n+p]^2 x^2 + [n+p]x,$$

and $\varphi_{n,3}(q_n; x) = [n+p]^3 x^3 + (2 + q_n)[n+p]^2 x^2 + T.L.P.(x)$.

Suppose that the relation is true for m , then we must prove it for $m + 1$.

By using(3.3), we have:

$$\varphi_{n,m+1}(q_n; x) = x \left\{ [m][n+p]^m x^{m-1} + \left(\sum_{j=0}^{m-1} (m-1-j) q_n^j \right) [m-1][n+p]^{m-1} x^{m-2} + T.L.P.(x) \right\} + [n+p]x \left\{ [n+p]^m x^m + \left(\sum_{j=0}^{m-1} (m-1-j) q_n^j \right) [n+p]^{m-1} x^{m-1} + T.L.P.(x) \right\}$$

$$\begin{aligned}
 &= [n + p]^{m+1}x^{m+1} + \left(\sum_{j=0}^{m-1} (m - 1 - j) q_n^j + [m] \right) [n + p]^m x^m + T.L.P. (x) \\
 &= [n + p]^{m+1}x^{m+1} + (m + (m - 1)q_n + \dots + 2q_n^{m-2} + q_n^{m-1})[n + p]^m x^m + T.L.P. (x) \\
 &= [n + p]^{m+1}x^{m+1} + \left(\sum_{j=0}^m (m - j) q_n^j \right) [n + p]^m x^m + T.L.P. (x). \quad \square
 \end{aligned}$$

Lemma 2: For $x \in [0, \infty)$, the following conditions are hold:

$$\begin{aligned}
 (i) W_n(1, q_n; x) &= 1; & (ii) W_n(t, q_n; x) &= \frac{q_n^n [p]}{[n]} x + x \rightarrow x \text{ as } n \rightarrow \infty; \\
 (iii) W_n(t^2, q_n; x) &= \left(1 + \frac{2q_n^n [p]}{[n]} + \frac{q_n^{2n} [p]^2}{[n]^2} \right) x^2 + \left(\frac{1}{[n]} + \frac{q_n^n [p]}{[n]^2} \right) x \rightarrow x^2 \text{ as } n \rightarrow \infty; \\
 (iv) W_n(t^3, q_n; x) &= \left(1 + \frac{3q_n^n [p]}{[n]} + \frac{3q_n^{2n} [p]^2}{[n]^2} + \frac{q_n^{3n} [p]^3}{[n]^3} \right) x^3 \\
 &\quad + \left(\frac{2 + q_n}{[n]} + \frac{(4 + 2q_n)q_n^n [p]}{[n]^2} + \frac{(2 + q_n)q_n^{2n} [p]^2}{[n]^3} \right) x^2 + \left(\frac{1}{[n]^2} + \frac{q_n^n [p]}{[n]^3} \right) x; \\
 (v) W_n(t^4, q_n; x) &= \left(1 + \frac{4q_n^n [p]}{[n]} + \frac{6q_n^{2n} [p]^2}{[n]^2} + \frac{4q_n^{3n} [p]^3}{[n]^3} + \frac{q_n^{4n} [p]^4}{[n]^4} \right) x^4 \\
 &\quad + \left(\frac{3 + 2q_n + q_n^2}{[n]} + \frac{(9 + 6q_n + 3q_n^2)q_n^n [p]}{[n]^2} + \frac{(9 + 6q_n + 3q_n^2)q_n^{2n} [p]^2}{[n]^3} \right. \\
 &\quad \left. + \frac{(3 + 2q_n + q_n^2)q_n^{3n} [p]^3}{[n]^4} \right) x^3 \\
 &\quad + \left(\frac{3 + 3q_n + q_n^2}{[n]^2} + \frac{(6 + 6q_n + 2q_n^2)q_n^n [p]}{[n]^3} + \frac{(3 + 3q_n + q_n^2)q_n^{2n} [p]^2}{[n]^4} \right) x^2 \\
 &\quad + \left(\frac{1}{[n]^3} + \frac{q_n^n [p]}{[n]^4} \right) x.
 \end{aligned}$$

Proof: From Lemma 1, we can prove (i).

We will prove the last one (v) and by the same way we prove {ii, iii, iv}.

Now; put $m = 3$ in (3.2), we get

$$\begin{aligned}
 W_n(t^4, q_n; x) &= \frac{x}{[n]} D_{q_n} \left(\left(1 + \frac{3q_n^n [p]}{[n]} + \frac{3q_n^{2n} [p]^2}{[n]^2} + \frac{q_n^{3n} [p]^3}{[n]^3} \right) x^3 \right. \\
 &\quad \left. + \left(\frac{2 + q_n}{[n]} + \frac{(4 + 2q_n)q_n^n [p]}{[n]^2} + \frac{(2 + q_n)q_n^{2n} [p]^2}{[n]^3} \right) x^2 + \left(\frac{1}{[n]^2} + \frac{q_n^n [p]}{[n]^3} \right) x \right)
 \end{aligned}$$

$$\begin{aligned}
 & + \frac{q^n[p]}{[n]} x \left(\left(1 + \frac{3q_n^n[p]}{[n]} + \frac{3q_n^{2n}[p]^2}{[n]^2} + \frac{q_n^{3n}[p]^3}{[n]^3} \right) x^3 \right. \\
 & + \left. \left(\frac{2 + q_n}{[n]} + \frac{(4 + 2q_n)q_n^n[p]}{[n]^2} + \frac{(2 + q_n)q_n^{2n}[p]^2}{[n]^3} \right) x^2 + \left(\frac{1}{[n]^2} + \frac{q_n^n[p]}{[n]^3} \right) x \right) \\
 & + x \left(\left(1 + \frac{3q_n^n[p]}{[n]} + \frac{3q_n^{2n}[p]^2}{[n]^2} + \frac{q_n^{3n}[p]^3}{[n]^3} \right) x^3 \right. \\
 & + \left. \left(\frac{2 + q_n}{[n]} + \frac{(4 + 2q_n)q_n^n[p]}{[n]^2} + \frac{(2 + q_n)q_n^{2n}[p]^2}{[n]^3} \right) x^2 + \left(\frac{1}{[n]^2} + \frac{q_n^n[p]}{[n]^3} \right) x \right) \\
 & = \left(\frac{1 + q_n + q_n^2}{[n]} + \frac{3[3]q_n^n[p]}{[n]^2} + \frac{3[3]q_n^{2n}[p]^2}{[n]^3} + \frac{[3]q_n^{3n}[p]^3}{[n]^4} \right) x^3 \\
 & + \left(\frac{[2](2 + q_n)}{[n]^2} + \frac{[2](4 + 2q_n)q_n^n[p]}{[n]^3} + \frac{[2](2 + q_n)q_n^{2n}[p]^2}{[n]^4} \right) x^2 \\
 & + \left(\frac{1}{[n]^3} + \frac{q_n^n[p]}{[n]^4} \right) x + \left(\frac{q_n^n[p]}{[n]} + \frac{3q_n^{2n}[p]^2}{[n]^2} + \frac{3q_n^{3n}[p]^3}{[n]^3} + \frac{q_n^{4n}[p]^4}{[n]^4} \right) x^4 \\
 & + \left(\frac{(2 + q_n)q_n^n[p]}{[n]^2} + \frac{(4 + 2q_n)q_n^{2n}[p]^2}{[n]^3} + \frac{(2 + q_n)q_n^{3n}[p]^3}{[n]^4} \right) x^3 \\
 & + \left(\frac{q_n^n[p]}{[n]^3} + \frac{q_n^{2n}[p]^2}{[n]^4} \right) x^2 + \left(1 + \frac{3q_n^n[p]}{[n]} + \frac{3q_n^{2n}[p]^2}{[n]^2} + \frac{q_n^{3n}[p]^3}{[n]^3} \right) x^4 \\
 & + \left(\frac{2 + q_n}{[n]} + \frac{(4 + 2q_n)q_n^n[p]}{[n]^2} + \frac{(2 + q_n)q_n^{2n}[p]^2}{[n]^3} \right) x^3 + \left(\frac{1}{[n]^2} + \frac{q_n^n[p]}{[n]^3} \right) x^2 \\
 & = \left(1 + \frac{4q_n^n[p]}{[n]} + \frac{6q_n^{2n}[p]^2}{[n]^2} + \frac{4q_n^{3n}[p]^3}{[n]^3} + \frac{q_n^{4n}[p]^4}{[n]^4} \right) x^4 \\
 & + \left(\frac{3 + 2q_n + q_n^2}{[n]} + \frac{(9 + 6q_n + 3q_n^2)q_n^n[p]}{[n]^2} + \frac{(9 + 6q_n + 3q_n^2)q_n^{2n}[p]^2}{[n]^3} \right. \\
 & + \left. \frac{(3 + 2q_n + q_n^2)q_n^{3n}[p]^3}{[n]^4} \right) x^3 \\
 & + \left(\frac{3 + 3q_n + q_n^2}{[n]^2} + \frac{(6 + 6q_n + 2q_n^2)q_n^n[p]}{[n]^3} + \frac{(3 + 3q_n + q_n^2)q_n^{2n}[p]^2}{[n]^4} \right) x^2 \\
 & + \left(\frac{1}{[n]^3} + \frac{q_n^n[p]}{[n]^4} \right) x \quad \square
 \end{aligned}$$

Theorem 1: (Korovkin Theorem for $W_n(f, q_n; x)$).

Suppose that $f \in C_\alpha[0, \infty)$ for some $\alpha > 0$ and f exists at a point $x \in (0, \infty)$, then

$$\lim_{n \rightarrow \infty} W_n(f, q_n; x) = f(x) \tag{3.5}$$

Further, if f exists and is continuous on $(a - \eta, b + \eta) \subset (0, \infty)$, $\eta > 0$, then (3.5) holds uniformly on $[a, b]$.

Proof: By using Lemma 2, the proof of the above theorem is held. \square

4. The q -moments of the operators $W_n(f, q_n; x)$.

Definition2: Form $\in N^0$ the m -th order q -moments $U_{n,m}(q_n; x)$ for the operators $W_n(f, q_n; x)$ are defined as:

$$U_{n,m}(q_n; x) = W_n((t - x)_{q_n}^m, q_n; x) = \sum_{k=0}^{\infty} Y_{n,k}(q_n; x) \left(\frac{[k]}{[n]} - x \right)_{q_n}^m$$

Lemma 3:For the function $U_{n,m}(q_n; x)$ we have:

$$\begin{aligned} (i) U_{n,0}(q_n; x) &= 1; & (ii) U_{n,1}(q_n; x) &= \frac{q_n^n [p]}{[n]} x; \\ (iii) U_{n,2}(q_n; x) &= \left(\frac{q_n^n [p]}{[n]} - \frac{q_n^{n+1} [p]}{[n]} + \frac{q_n^{2n} [p]^2}{[n]^2} \right) x^2 + \left(\frac{1}{[n]} + \frac{q_n^n [p]}{[n]^2} \right) x; \\ (iv) U_{n,3}(q_n; x) &= \left(\frac{(1 - q_n - q_n^2 + q_n^3) q_n^n [p]}{[n]} + \frac{(2 - q_n - q_n^2) q_n^{2n} [p]^2}{[n]^2} + \frac{q_n^{3n} [p]^3}{[n]^3} \right) x^3 \\ &+ \left(\frac{1 - q_n^2}{[n]} + \frac{(3 + q_n - q_n^2) q_n^n [p]}{[n]^2} + \frac{(2 + q_n) q_n^{2n} [p]^2}{[n]^3} \right) x^2 + \left(\frac{1}{[n]^2} + \frac{q_n^n [p]}{[n]^3} \right) x; \\ (v) U_{n,4}(q_n; x) &= \left(\frac{(1 - q_n - q_n^2 + q_n^4 + q_n^5 - q_n^6) q_n^n [p]}{[n]} + \frac{(3 - 2q_n - 2q_n^2 - q_n^3 + q_n^4 + q_n^5) q_n^{2n} [p]^2}{[n]^2} \right. \\ &+ \left. \frac{(3 - q_n - q_n^2 - q_n^3) q_n^{3n} [p]^3}{[n]^3} \right) x^4 \\ &+ \left(\frac{1 + q_n - q_n^2 - q_n^3 + q_n^5}{[n]} + \frac{(5 + q_n - 2q_n^2 - 4q_n^3 - q_n^4 + q_n^5) q_n^n [p]}{[n]^2} \right. \\ &+ \left. \frac{(7 + 3q_n - 3q_n^3 - q_n^4) q_n^{2n} [p]^2}{[n]^3} + \frac{(3 + 2q_n + q_n^3) q_n^{3n} [p]^3}{[n]^4} \right) x^3 \\ &+ \left(\frac{2 + 2q_n + q_n^3}{[n]^2} + \frac{(5 + 5q_n + q_n^2 - q_n^3) q_n^n [p]}{[n]^3} + \frac{(3 + 3q_n + q_n^2) q_n^{2n} [p]^2}{[n]^4} \right) x^2 \\ &+ \left(\frac{1}{[n]^3} + \frac{q_n^n [p]}{[n]^4} \right) x. \end{aligned}$$

Proof: we will prove (v) and by the same way we can prove {i, ii, iii, iv},

$$\begin{aligned} (v) U_{n,4}(q_n; x) &= W_n((t - x)_{q_n}^4, q_n; x) = W_n(t^4, q_n; x) - [4]x W_n(t^3, q_n; x) \\ &+ (q_n[5] + q_n^3)x^2 W_n(t^2, q_n; x) - q_n^3[4]x^3 W_n(t, q_n; x) + q_n^6 x^4 W_n(1, q_n; x) \end{aligned}$$

By using Lemma 2, we get

$$\begin{aligned}
 U_{n,4}(q_n; x) &= \left(1 + \frac{4q_n^n[\square]}{[n]} + \frac{6q_n^{2n}[p]^2}{[n]^2} + \frac{4q_n^{3n}[p]^3}{[n]^3} + \frac{q_n^{4n}[p]^4}{[n]^4} - [4] - \frac{[4]3q_n^n[p]}{[n]} - \frac{[4]3q_n^{2n}[p]^2}{[n]^2} \right. \\
 &\quad - \frac{[4]q_n^{3n}[p]^3}{[n]^3} + (q_n[5] + q_n^3) + \frac{(q_n[5] + q_n^3)2q_n^n[p]}{[n]} + \frac{(q_n[5] + q_n^3)q_n^{2n}[p]^2}{[n]^2} \\
 &\quad \left. - q_n^3[4] - \frac{q_n^3[4]q_n^n[p]}{[n]} + q_n^6 \right) x^4 \\
 &\quad + \left(\frac{3 + 2q_n + q_n^2}{[n]} + \frac{(9 + 6q_n + 3q_n^2)q_n^n[p]}{[n]^2} + \frac{(9 + 6q_n + 3q_n^2)q_n^{2n}[p]^2}{[n]^3} \right. \\
 &\quad + \frac{(3 + 2q_n + q_n^2)q_n^{3n}[p]^3}{[n]^4} - \frac{[4](2 + q_n)}{[n]} - \frac{[4](4 + 2q_n)q_n^n[p]}{[n]^2} \\
 &\quad \left. - \frac{[4](2 + q_n)q_n^{2n}[p]^2}{[n]^3} + \frac{(q_n[5] + q_n^3)}{[n]} + \frac{(q_n[5] + q_n^3)q_n^n[p]}{[n]^2} \right) x^3 \\
 &\quad + \left(\frac{3 + 3q_n + q_n^2}{[n]^2} + \frac{(6 + 6q_n + 2q_n^2)q_n^n[p]}{[n]^3} + \frac{(3 + 3q_n + q_n^2)q_n^{2n}[p]^2}{[n]^4} - \frac{[4]}{[n]^2} \right. \\
 &\quad \left. - \frac{[4]q_n^n[p]}{[n]^3} \right) x^2 + \left(\frac{1}{[n]^3} + \frac{q_n^n[p]}{[n]^4} \right) x \\
 &= \left(\frac{(1 - q_n - q_n^2 + q_n^4 + q_n^5 - q_n^6)q_n^n[p]}{[n]} + \frac{(3 - 2q_n - 2q_n^2 - q_n^3 + q_n^4 + q_n^5)q_n^{2n}[p]^2}{[n]^2} \right. \\
 &\quad \left. + \frac{(3 - q_n - q_n^2 - q_n^3)q_n^{3n}[p]^3}{[n]^3} \right) x^4 \\
 &\quad + \left(\frac{1 + q_n - q_n^2 - q_n^3 + q_n^5}{[n]} + \frac{(5 + q_n - 2q_n^2 - 4q_n^3 - q_n^4 + q_n^5)q_n^n[p]}{[n]^2} \right. \\
 &\quad \left. + \frac{(7 + 3q_n - 3q_n^3 - q_n^4)q_n^{2n}[p]^2}{[n]^3} + \frac{(3 + 2q_n + q_n^3)q_n^{3n}[p]^3}{[n]^4} \right) x^3 \\
 &\quad + \left(\frac{2 + 2q_n + q_n^3}{[n]^2} + \frac{(5 + 5q_n + q_n^2 - q_n^3)q_n^n[p]}{[n]^3} + \frac{(3 + 3q_n + q_n^2)q_n^{2n}[p]^2}{[n]^4} \right) x^2 \\
 &\quad + \left(\frac{1}{[n]^3} + \frac{q_n^n[p]}{[n]^4} \right) x. \quad \square
 \end{aligned}$$

5. The recurrence relation for the q-moments of the operators $W_n(f, q_n; x)$.

In this section we give some lemmas which help us to prove the Voronovskaja-type asymptotic formula for the operators $W_n(f, q_n; x)$.

Lemma 4: Form $\in N^0, p \geq 0$,

$$\begin{aligned}
 [n]U_{n,m+1}(q_n; x) &= xD_{q_n}U_{n,m}(q_n; x) + q_n^n[p]xU_{n,m}(q_n; x) + [n](1 - q_n^m)xU_{n,m}(q_n; x) \\
 &\quad + [m]xU_{n,m-1}(q_n; q_nx). \tag{5.1}
 \end{aligned}$$

Furthermore

1. $U_{n,m}(q_n; x)$ is a polynomial in x of degree m .

2. For every $x \in [0, \infty)$, $U_{n,m}(q_n; x) = O([n]^{-(m)})$.

Proof: By using Lemma 1, we get

$$\begin{aligned}
 D_{q_n} U_{n,m}(q_n; x) &= \sum_{k=0}^{\infty} -[m] \left(\frac{[k]}{[n]} - q_n x \right)_{q_n}^{m-1} Y_{n,k}(q_n; q_n x) + \sum_{k=0}^{\infty} \left(\frac{[k]}{[n]} - x \right)_{q_n}^m D_{q_n} Y_{n,k}(q_n; x) \\
 x D_{q_n} U_{n,m}(q_n; x) &= -[m] x U_{n,m-1}(q_n; q_n x) + \sum_{k=0}^{\infty} Y_{n,k}(q_n; x) ([k] - [n]x - q_n^n [p]x) \left(\frac{[k]}{[n]} - x \right)_{q_n}^m \\
 &= -[m] x U_{n,m-1}(q_n; q_n x) + [n] \sum_{k=0}^{\infty} Y_{n,k}(q_n; x) \left(\frac{[k]}{[n]} - q_n^m x \right) \left(\frac{[k]}{[n]} - x \right)_{q_n}^m \\
 &\quad - (1 - q_n^m) [n] x \sum_{k=0}^{\infty} Y_{n,k}(q_n; x) \left(\frac{[k]}{[n]} - x \right)_{q_n}^m - q_n^n [p] x U_{n,m}(q_n; x) \\
 &= -q_n^n [p] x U_{n,m}(q_n; x) - [m] x U_{n,m-1}(q_n; q_n x) + [n] U_{n,m+1}(q_n; x) \\
 &\quad - (1 - q_n^m) [n] x U_{n,m}(q_n; x).
 \end{aligned}$$

Now, The consequence (1) can be proven easily by using (5.1) and the induction on m , so the details are omitted.

Now, we prove (2) by using induction on m .

The consequence is true for $m = 1$.

Now, let the consequence be true for m , we must prove that the consequence is true for $m + 1$.

$$\begin{aligned}
 [n] U_{n,m+1}(q_n; x) &= O([n]^{-(m)}) + O([n]^{-(m)}) + O([n]^{-(m-1)}) + O([n]^{-(m-1)}) \\
 &= O([n]^{-(m)}) + O([n]^{-(m-1)})
 \end{aligned}$$

$$U_{n,m+1}(q_n; x) = O([n]^{-(m+1)}) + O([n]^{-(m)})$$

$$U_{n,m+1}(q_n; x) = O([n]^{-(m+1)}), \text{ so the result is true for } (m + 1).$$

Hence, $U_{n,m}(q_n; x) = O([n]^{-(m)})$ for every $x \in [0, \infty)$. □

Lemma 5: Let δ and α be any two positive real numbers and $[a, b] \subset (0, \infty)$.

Then, for any $s > 0$ we have:

$$\sup_{x \in [a,b]} \left| \sum_{\left| \frac{[k]}{[n]} - x \right| \geq \delta} Y_{n,k}(q; x) e_{q_n}^{\alpha t} \right| = O([n]^{-s}).$$

Making use of q -Taylor's expansion, Schwarz inequality for summation and Lemma 4(2), the proof of this lemma easily follows, hence the details are omitted.

6. Voronovaskaja – type asymptotic formula for $W_n(f, q_n; x)$.

The next theorem is a Voronovskaja-type asymptotic formula for the operators $W_n(f(t), q_n; x)$;

Theorem 2: Let $f \in C_\alpha[0, \infty)$ for some $\alpha > 0$. If $D_{q_n}f, D_{q_n}^2f$ are exist at a point $x \in (0, \infty)$, then for sufficiently large n we have:

$$\lim_{n \rightarrow \infty} [n] (W_n(f(t), q_n; x) - f(x)) = \frac{x}{[2]!} D_{q_n}^2 f(x)$$

Proof: By using q -Taylor's expansion of f , we get:

$$f(t) = f(x) + \frac{(D_{q_n}f(x))}{[1]!} (t-x)_{q_n} + \frac{(D_{q_n}^2f(x))}{[2]!} (t-x)_{q_n}^2 + r(t; x)(t-x)_{q_n}^2$$

where $r(t; x) \rightarrow 0$ as $t \rightarrow x$.

Then;

$$W_n(f(t), q_n; x)$$

$$\begin{aligned} &= f(x)W_n(1, q_n; x) + \frac{(D_{q_n}f(x))}{[1]!} W_n((t-x)_{q_n}, q_n; x) \\ &+ \frac{(D_{q_n}^2f(x))}{[2]!} W_n((t-x)_{q_n}^2, q_n; x) + W_n(r(t; x)(t-x)_{q_n}^2, q_n; x) \\ &= f(x) + (D_{q_n}f(x)) U_{n,1}(q_n; x) + \frac{(D_{q_n}^2f(x))}{[2]!} U_{n,2}(q_n; x) \\ &\quad + \sum_{k=0}^{\infty} Y_{n,k}(q_n; x) \left(\frac{[k]}{[n]} - x\right)_{q_n}^2 \varepsilon \left(\frac{[k]}{[n]}; x\right). \end{aligned} \tag{6.1}$$

Suppose that

$$J := \sum_{k=0}^{\infty} Y_{n,k}(q_n; x) \left(\frac{[k]}{[n]} - x\right)_{q_n}^2 \varepsilon \left(\frac{[k]}{[n]}; x\right).$$

Since $r(t; x) \rightarrow 0$ as $t \rightarrow x$, for a given $\varepsilon > 0$ there exists $\delta > 0$ such that $\left| r\left(\frac{[k]}{[n]}; x\right) \right| < \varepsilon$,

whenever $0 < \left| \frac{[k]}{[n]} - x \right| < \delta$, for $\left| \frac{[k]}{[n]} - x \right| \geq \delta$, there exists a constant $C > 0$ such that

$$\left| \left(\frac{[k]}{[n]} - x\right)_{q_n}^2 r\left(\frac{[k]}{[n]}; x\right) \right| \leq C e_{q_n}^{\alpha\left(\frac{[k]}{[n]}\right)}, \text{ hence;}$$

$$\begin{aligned} |J| &\leq \left| \sum_{\left|\frac{[k]}{[n]} - x\right| < \delta} Y_{n,k}(q_n; x) \left(\frac{[k]}{[n]} - x\right)_{q_n}^2 r\left(\frac{[k]}{[n]}; x\right) \right| + \left| \sum_{\left|\frac{[k]}{[n]} - x\right| \geq \delta} Y_{n,k}(q_n; x) C e_{q_n}^{\alpha\left(\frac{[k]}{[n]}\right)} \right| \\ &:= l_1 + l_2. \end{aligned}$$

Then, by using Schwarz inequality for summation, we get:

$$l_1 \leq \varepsilon \left(\sum_{k=0}^{\infty} Y_{n,k}(q_n; x) \right)^{1/2} \left(\sum_{k=0}^{\infty} Y_{n,k}(q_n; x) \left(\frac{[k]}{[n]} - x\right)_{q_n}^4 \right)^{1/2}.$$

In view of Lemma 4 and since ε is arbitrary, then,

$$l_1 \leq \varepsilon \left(U_{n,4}(q_n; x) \right)^{1/2} = \varepsilon O([n]^{-2}) = O(1) = o(1).$$

From Lemma 5 we have: $l_2 = C O([n]^{-s}) = O([n]^{-s})$.

$$|J| = o(1), \text{ as } n \rightarrow \infty.$$

Then;

$$\lim_{n \rightarrow \infty} \sum_{k=0}^{\infty} Y_{n,k}(q_n; x) \left(\frac{[k]}{[n]} - x \right)_{q_n}^2 r \left(\frac{[k]}{[n]}; x \right) = 0.$$

By applying this result in (6.1), we get:

$$\lim_{n \rightarrow \infty} [n] \left(W_n(f(t), q_n; x) - f(x) \right) = \frac{x}{[2]!} D_{q_n}^2 f(x).$$

To prove the uniformity assertion, it is sufficient to remark that $\delta(\varepsilon)$ in the above proof can be chosen to be independent of $x \in [a, b]$, and also the other estimates hold uniformly in $x \in [a, b]$,

The proof is completed \square

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المستخلص

في هذا البحث ، نقدم وندرس نمط جديد من مؤثرات q -Szász-Mirakjan $W_n(f, q_n; x)$ في الحالة $q_n \in (0,1)$. في البداية نبرهن إن هذه المؤثرات تتقارب إلى الداله المستخدمة في التقريب ثم سنعرف العزوم من الرتبه m ومن النمط q ونشتق الصيغة التكرارية لهذه العزوم ، وأخيراً نبرهن الصيغة المشابهه لصيغة Voronovskaja للمؤثرات $W_n(f, q_n; x)$.