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Development treatment of initial boundary value problems for one dimensional heat-like and wave-like equations using homotopy perturbation method

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Abstract

In this paper, a new technique is applied to develop the treatment of initial boundary value problems for one dimensional heat-like and wave-like partial differential equations [3] (ordinary or fractional) by mixed initial and boundary conditions together to obtain a new initial solution at every iteration using homotopy perturbation method (HPM). The structure of a new successive initial solutions can give a more accurate solution in a first step.

Keywords: initial boundary value problems, one dimensional, heat-like and wave-like partial differential equations, homotopy perturbation method.

1. Introudaction

Many researchers discussed the initial and boundary value problems such as [4,10,11,13,15,16,1718] Although the problems of these researches contain initial boundary value problems, but the researchers discussed those problems by using either initial or boundary conditions. So we present a reliable framework by applying a new technique for treatment initial and boundary value problems by mixed initial conditions with boundary conditions together to obtain a new initial solution at every iteration using homotopy perturbation method. Such as technique was applied by[1,2] for treatment of initial boundary value problems. In this paper, a new technique is applied to develop the treatment of initial boundary value problems for one dimensional heat-like and wave-like partial differential equations (ordinary or fractional) by construct a new successive initial solutions using homotopy perturbation method, which can give a more accurate solution, some examples are given to illustrate the effectiveness and convenience of this technique.

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 1.1. Jumarie (2009) is defined the fractional derivative as the following limit form

$$f^{(\alpha)} = \lim_{h\to 0} \frac{\Delta^\alpha [f(x)-f(0)]}{h^\alpha}. \quad (1.1)$$

This definition is close to the standard definition of derivatives, and as a direct result, the α^{th} derivative of a constant, $0 < \alpha < 1$ is zero.

Definition 1.2. Fractional integral operator of order $\alpha \geq 0$ is defined as

$$I_x^\alpha f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau)d\tau, \quad \alpha > 0, \qquad (1.2)$$

where Γ is a gamma function.

Definition 1.3. Fractional derivative of $f(x)$ in the [Caputo, 1967] sense is defined as

$$D_x^\alpha f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\tau)^{m-\alpha-1} \frac{d^m f(\tau)}{d\tau^m} d\tau, m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0. \qquad (1.3)$$

α is the order of the derivative. For the Caputo's derivative we have:

$1 - D^\alpha C = 0, \quad C$ is constant, (1.4)

$2 - D^\alpha x^\beta = 0, \quad \beta \leq \alpha - 1,$ (1.5)

$3 - D^\alpha x^\beta = \frac{\Gamma(1+\beta)}{\Gamma(1-\alpha+\beta)} x^{\beta-\alpha}, \quad \beta > \alpha - 1,$ (1.6)

Definition 1.4. Fractional derivative of compounded functions [Jumarie, 2009] is defined as

$$d^\alpha f \cong \Gamma(1+\alpha)df, \quad 0 < \alpha < 1. \qquad (1.7)$$

Definition 1.5. The integral with respect to $(dx)^\alpha$ [Jumarie, 2009] is defined as the solution of the fractional differential equation

$$dy \cong f(x)(dx)^\alpha, \quad x \geq 0, \quad y(0) = 0, \quad 0 < \alpha < 1. \qquad (1.8)$$

Lemma 2.1. Let $f(x)$ denote a continuous function [Jumarie, 2009] then the solution of the Eq. (1.5) is defined as

$$y = \int_0^x f(\tau)(d\tau)^\alpha = \alpha \int_0^x (x-\tau)^{\alpha-1} f(\tau)d\tau, \quad 0 < \alpha < 1. \qquad (1.9)$$

For example $f(x) = x^\alpha$ in Eq. (2.6) one obtain

$$\int_0^x \tau^\gamma (d\tau)^\alpha = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \quad 0 < \alpha \leq 1.$$
(1.10)

2-Homotopy perturbation method.

Consider the following nonlinear differential equation;

$$A(u) = f(r) \quad r \in \Omega, \tag{2.1}$$

with boundary conditions:

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{2.2}$$

where A is a general differential operator, B is a boundary operator, $f(r)$is a known analytic function, Γ is the boundary of the domain Ω,The operator A can be generally divided into two parts L, and N, where L is linear, and N, is nonlinear, therefore equation.(2.1) can be written as,

$$L(u) + N(u) = f(r), \tag{2.3}$$

By using homotopy technique[7,8], one can construct a homotopy $v(r,p): \Omega \times [0,1] \to R$ which

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \; p \in [0,1], \tag{2.4}$$

or

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{2.5}$$

where $r \in \Omega$ and $p \in [0,1]$ is an embedding parameter, and u_0 is the initial approximation of equation.(2.3)which satisfies the boundary conditions. Hence, obviously we have

$$H(v,0) = L(v) - L(u_0) = 0, \tag{2.6}$$

$$H(v,1) = A(v) - f(r) = 0, \tag{2.7}$$

and the changing process of p from 0 to 1 is the same as changing $H(v,p)$ from $L(v) - L(u_0)$ to $A(v) - f(r)$. In topology, this is called deformation, and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic. If, the embedding parameter p; is considered as a small parameter, applying the classical perturbation technique, we can assume that the solution of equation.(2.7) can be given as a power series in p, i.e.

$$v = v_0 + pv_1 + p^2 v_2 + \cdots \tag{2.8}$$

and setting $p = 1$ results in the approximate solution of equation.(2.3) as ;

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots \tag{2.9}$$

3. New technique for solving one dimensional heat-like and wave-like equations (ordinary or fractional) using HPM

To convey the basic idea for modified treatment of initial boundary value problems by homotopy perturbation to solve one dimensional heat-like and wave-like equations of the form

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t)$$

the initial conditions associated with Eq. (3.1) are of the form

$$u(x,0) = f_0(x), \; \frac{\partial u(x,0)}{\partial t} = f_1(x), \; , \quad 0 < x < 1, \tag{3.2}$$

and the boundary conditions are given by

$$u(0,t) = g_0(t), \; u(1,t) = g_1(t), \quad t > 0, \tag{3.3}$$

where $f_0(x), f_1(x), g_0(t)$ and $g_1(t)$ are given functions. The initial solution can be written as $u_0(x,t) = f_0(x) + tf_1(x)$.

The initial values are usually used for selecting the zeroth approximation u_0 but in this paper we accredit a new technique to calculate the zeroth approximation u_0^* by construct a new initial solutions u_n^* by mixed initial conditions in Eq. (3.2) with boundary conditions in Eq. (3.3) at every iteration as follows [1,2]

$u_n^*(x,t) = u_n(x,t) + (1-x)[g_0(t) - u_n(0,t)] + x[g_1(t) - u_n(1,t)],\ \ n \geq 0. \qquad (3.4)$

It is obvious that the new successive initial solutions u_n^* in Eq. (3.4) satisfying the initial and boundary conditions together as follows

$$if\ \ x = 0\ \ then\ \ u_n^*(0,t) = g_0(t),$$

$$if\ \ x = 1\ \ then\ \ u_n^*(1,t) = g_1(t),$$

$$if\ \ t = 0\ \ then\ \ u_n^*(x,0) = u_n(x,0). \qquad (3.5)$$

The second and third terms in right side of Eq. (3.4) will be vanish when we applying the second derivative by x which was appearing in a right side of Eq. (3.1), so to establish these terms we can be modified Eq. (3.4) and rewritten in a new formulation as

$$u_n^*(x,t) = u_n(x,t) + (1-x^2)[g_0(t) - u_n(0,t)] + x^2[g_1(t) - u_n(1,t)], n \geq 0. \qquad (3.6)$$

Such as treatment is a very effective as shown in this paper.

4. Applications and results

Example 1[13]: Consider the following one-dimensional heat-like problem

$$\frac{\partial u}{\partial t} - \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} = 0,\ \ 0 < x < 1, t > \ 0, \qquad (4.1)$$

subject to the initial conditions

$$u(x,0) = x^2,\ \ 0 < x < 1, \qquad (4.2)$$

and the boundary conditions

$$u(0,t) = 0\,,\quad u(1,t) = \mathrm{e}^{t}\,,\quad t > 0, \qquad (4.3)$$

By applying a new approximations u_n^* in Eq. (3.6) we obtain

$$u_n^*(x,t) = u_n(x,t) + (1-x^2)[0 - u_n(0,t)] + x^2[\mathrm{e}^{t} - u_n(1,t)], \qquad (4.4)$$

Now, we begin with a new initial approximation u_0^* $(when\ n = 0)$

$$u_0^*(x,t) = x^2\ \mathrm{e}^{t}, \qquad (4.5)$$

we construct the following homotopy

$$(1-p)\left(\frac{\partial u}{\partial t} - \frac{\partial u_0}{\partial t}\right) + p\left(\frac{\partial u}{\partial t} - \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\right) = 0 \qquad (4.6)$$

where the initial approximation $u_0(x,0) = x^2$,

$$\frac{\partial u}{\partial t} + p\left(-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\right) = 0, \qquad (4.7)$$

we consider u as

$$u = u_0 + pu_1 + p^2u_2 + \cdots \qquad (4.8)$$

substituting (4.8) in (4.7) and equating the coefficients of powers of p, we get following set of differential equations with solving the systems accordingly, we obtain,

$$p^0{:}\frac{\partial u}{\partial t} = 0, u_0(x,0) = x^2 \Rightarrow u_0(x,t) = x^2$$

$$p^1 : \frac{\partial u_1}{\partial t} = \frac{1}{2}x^2\frac{\partial u_0^*}{\partial x^2} \Rightarrow u_1 = -x^2 + x^2e^t$$

through, the solution of two steps iteration and by setting $p = 1$ in equation.(4.8), the solution of (4.1) can be obtain as

$$u(x,t) = u_0(x,t) + u_1(x,t)$$

$u(x,t) = x^2 e^t$, which is the exact solution.

Table.1.Approxamation(analytic) solutions, t=1.

x	Exact solution	ADM n=10	VIM n=10	HPM(1) n=10	HPM(2) n=1
0	0.00000000000	0.00000000000	0.00000000000	0.00000000000	0.00000000000
0.1	0.02718281828	0.02718281803	0.02718281803	0.02718281803	0.02718281828
0.2	0.10873127310	0.1087312721	0.10873127210	0.10873127210	0.10873127310
0.3	0.24464536450	0.24464536230	0.24464536230	0.24464536230	0.24464536450
0.4	0.43492509250	0.43492508850	0.43492508850	0.43492508850	0.43492509250
0.5	0.67957045700	0.67957045080	0.67957045080	0.67957045080	0.67957045700
0.6	0.97858145810	0.97858144910	0.97858144910	0.97858144910	0.97858145810
0.7	1.33195809600	1.33195808300	1.33195808300	1.33195808300	1.33195809600
0.8	1.73970037000	1.73970035400	1.73970035400	1.73970035400	1.73970037000
0.9	2.20180828100	2.20180826000	2.20180826000	2.20180826000	2.20180828100
1	2.71828182800	2.71828180300	2.71828180300	2.71828180300	2.71828182800

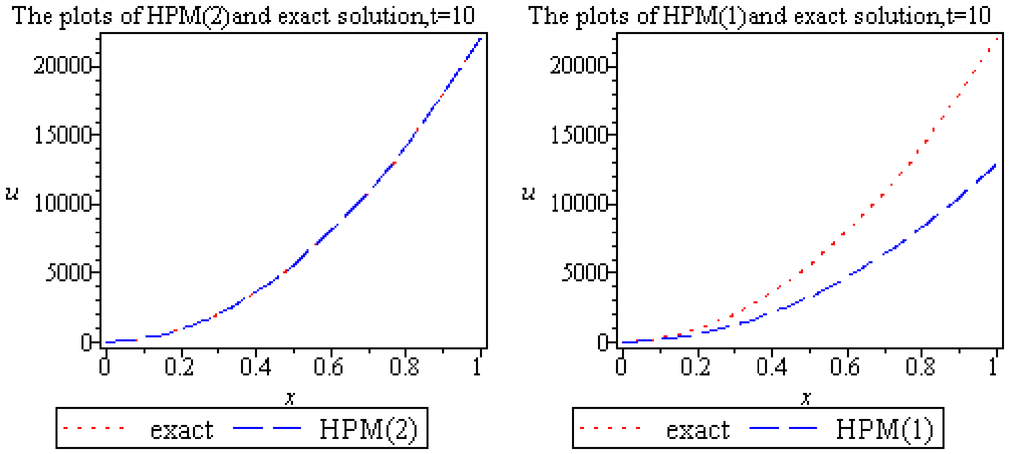

Fig.1. Where ADM is Adomian Decomposition Method in [1] , VIM is Variation Iteration Method in [2], HPM(1) is standard Homtopy Perturbation Method and HPM(2) is a new technique using Homotopy Perturbation Method.

***Example 2*[13]:** We next consider the one-dimensional wave-like equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} = 0,$$

$$0 < x < 1, t > 0, \tag{4.10}$$

subject to the initial conditions

$$u(x,0) = x, \quad \frac{\partial u(x,0)}{\partial t} = x^2, \quad 0 < x < 1, \tag{4.11}$$

and the boundary conditions

$$u(0,t) = 0 \quad , \quad u(1,t) = 1 + \sinh t, \quad t > 0, \tag{4.12}$$

By applying a new approximations u_n^* in Eq. (3.6) we have

$$u_n^*(x,t) = u_n(x,t) + (1 - x^2)[0 - u_n(0,t)] + x^2[1 + \sinh t - u_n(1,t)], \tag{4.13}$$

where $n = 0,1,2, \dots$. The initial approximation is $u_0(x,t) = x + x^2 t$.

Now, we begin with a new initial approximation u_0^* $(when\ n = 0)$

$$u_0^*(x,t) = x + x^2 \sinh t. \tag{4.14}$$

we construct the following homotopy

$$(1-p)\left(\frac{\partial^2 u}{\partial t^2}-\frac{\partial^2 u_0}{\partial t^2}\right)+p\left(\frac{\partial^2 u}{\partial t^2}-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\right)=0 \qquad (4.15)$$

where the initial approximation is $u_0(x,0)=x$,

$$\frac{\partial^2 u}{\partial t^2}+p\left(-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\right)=0, \qquad (4.16)$$

we consider u as

$$u=u_0+pu_1+p^2u_3+\cdots \qquad (4.17)$$

substituting (4.17) in (4.16) and equating the coefficients of powers p, we get following set of differential equations with solving the systems accordingly, we obtain,

$$p^0:\frac{\partial^2 u}{\partial t^2}=0, u_0(x,0)=x, \Rightarrow u_0(x,t)=x+x^2t$$

$$p^1:\frac{\partial^2 u_1}{\partial t^2}=\frac{1}{2}x^2\frac{\partial^2 u_0^*}{\partial x^2}\Rightarrow u_1=x-x^2t+x^2t+x^2 sinht$$

by setting $p=1$ in equation.(4.17), the solution of (4.10) can be obtain as

$$u(x,t)=u_0(x,t)+u_1(x,t)$$

$u(x,t)=x+x^2\sinh t$. Which is the exact solution. $\qquad (4.18)$

Table.2. Approximation(analytic)solutions, t=10

x	Exact solution	ADM n=10	VIM n=10	HPM(1) n=10	HPM(2) n=1
0	0.000000000	0.000000000	0.000000000	0.000000000	0.000000000
0.1	110.2323287	110.1861552	110.1861552	110.1861552	110.2323287
0.2	440.7293148	440.5446208	440.5446208	440.5446208	440.7293148
0.3	991.4909583	991.0753968	991.0753968	991.0753968	991.4909583
0.4	1762.517259	1761.778483	1761.778483	1761.778483	1762.517259
0.5	2753.808218	2752.653880	2752.653880	2752.653880	2753.808218
0.6	3965.363833	3963.701587	3963.701587	3963.701587	3965.363833
0.7	5397.184106	5394.921605	5394.921605	5394.921605	5397.184106
0.8	7049.269037	7046.313933	7046.313933	7046.313933	7049.269037
0.9	8921.618625	8917.878571	8917.878571	8917.878571	8921.618625
1	11014.23287	11009.61552	11009.61552	11009.61552	11014.23287

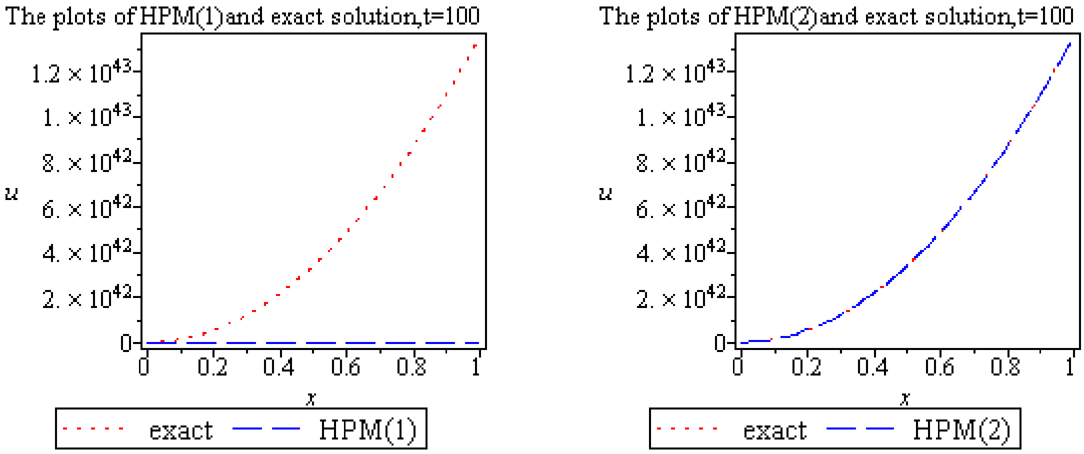

Fig.2. Where ADM is Adomian Decomposition Method in [1] , VIM is Variation Iteration Method in [2], HPM(1) is standard Homotopy Perturbation Method and HPM(2) is Homotopy Perturbation Method in this paper.

***Example 3*[13,18]:** Consider the following one-dimensional fractional heat-like problem

$$\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{1}{2}x^2\frac{\partial^2}{\partial x}$$

subject to the initial condition

$$u(x,0) = x^2,\ 0 < x < 1, \tag{4.20}$$

and the boundary conditions

$$u(0,t) = 0,\quad u(1,t) = e^t,\quad t > 0, \tag{4.21}$$

the exact solution is $u(x,t) = x^2 E_\alpha(t^\alpha)$, (4.22)

where

$\mathrm{E}_\alpha(\mathrm{t}^\alpha) = \lim_{m\to\infty} \sum_{k=0}^{m} \frac{t^{k\alpha}}{\Gamma(1+k\alpha)}$.

By applying a new approximations u_n^* in Eq. (3.6) we have

$u_n^*(x,t) = u_n(x,t) + (1-x^2)[0 - u_n(0,t)] + x^2[\mathrm{E}_\alpha(\mathrm{t}^\alpha) - u_n(1,t)]$,

(4.23)

where $n = 0,1,2,\dots$. The initial approximation is $u_0(x,t) = x^2$.

To begin with a new initial approximation u_0^* we applying Eq. (4.13) at $n = 0$ such as

$$u_0^*(x,t) = x^2 e^t$$

$$= x^2\left(1 + \frac{t}{\Gamma(2)} + \frac{t^2}{\Gamma(3)} + \cdots\right). \tag{4.24}$$

we construct the following homotopy

$$(1-p)\left(\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha}\right) + p\left(\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\right) = 0, \tag{4.25}$$

where the initial approximation $u_0(x,0) = x^2$,

$$\frac{\partial^\alpha u}{\partial t^\alpha} + p\left(-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\right) = 0, \tag{4.26}$$

we consider u as

$$u = u_0 + pu_1 + p^2u_3 + \cdots, \tag{4.27}$$

substituting (4.27) in (4.26) and equating the coefficients of powers p, we get following set of differential equations with solving the systems accordingly, we obtain,

$p^0: \frac{\partial^\alpha u}{\partial t^\alpha} = 0, by\ Eq.(4.20), Eq.(1.4), we\ have\ u_0(x,t) = x^2,$

$$p^1: \frac{\partial^\alpha u_1}{\partial t^\alpha} = \frac{1}{2}x^2\frac{\partial^2 u_0^*}{\partial x^2} \Rightarrow\ u_1 = \frac{x^2}{2\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}\frac{\partial^2(x^2e^\xi)}{\partial x^2}d\xi$$

$$= \frac{x^2}{\Gamma(\alpha)}\int_0^t (t-\xi)^{\alpha-1}e^\xi\, d\xi$$

$$= \frac{x^2}{\Gamma(\alpha)}\int_0^t ((t-\xi)^{\alpha-1}(1 + \frac{\xi}{\Gamma(2)} + \frac{\xi^2}{\Gamma(3)} + \cdots)d\xi,$$

$$= \frac{x^2}{\Gamma(\alpha)}(\mathcal{B}(\alpha,1)t^\alpha + \frac{\mathcal{B}(\alpha,2)t^{\alpha+1}}{\Gamma(2)} + \frac{\mathcal{B}(\alpha,3)t^{\alpha+2}}{\Gamma(3)} + \cdots),$$

$$= x^2\left(\frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{t^{\alpha+2}}{\Gamma(3+\alpha)} + \cdots\right),$$

where $\mathcal{B}$is beta function.

Through, the solution of two steps iteration and by setting $p = 1$ in equation.(4.27), the solution of (4.19) can be obtain as

$$u(x,t) = u_0(x,t) + u_1(x,t),$$

$$u(x,t) = x^2\left(1 + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{\alpha+1}}{\Gamma(2+\alpha)} + \cdots\right), \tag{4.28}$$

Let $\alpha = 1$, Eq.(4.24) becomes

$$\mathrm{u(x,t)} = \mathrm{x}^2\left(1 + \mathrm{t} + \frac{\mathrm{t}^2}{2!} + \cdots\right) = \mathbf{x^2e^t}, \tag{4.29}$$

which yields an exact solution of Eq. (4.15) when $\alpha = 1$. But, generally the solution (4.29) is not exactly for Eq. (4.19), because it doesn’t satisfying of vinery equation. In the other hand the boundary conditions (4.21) is not corresponding to the exact solution (4.22) which were given in [13,18], that's mean the boundary conditions (4.21) are be not true for this problem. So, in the following example we present a reliable framework by applying another boundary conditions which can be satisfying by exact solution(4.22).

Table.3. Approximation(analytic)solutions, x=0.9

t	Exact solution	VIM	ADM	HPM(1)	HPM(2)
0	0.810000000	0.810000000	0.810000000	0.810000000	0.810000000
1	2.201808281	2.201808260	2.201808260	2.201808260	2.201808281
2	5.985135440	5.985085714	5.985085714	5.985085714	5.985135440
3	16.26928491	16.26452879	16.26452879	16.26452879	16.26928491
4	44.22450152	44.09891430	44.09891430	44.09891430	44.22450152
5	120.2146589	118.5682869	118.5682869	118.5682869	120.2146589
6	326.7773227	312.8497714	312.8497714	312.8497714	326.7773227
7	888.2728580	800.7595110	800.7595110	800.7595110	888.2728580
8	2414.575969	1970.018229	1970.018229	1970.018229	2414.575969
9	6563.497982	4633.752916	4633.752916	4633.752916	6563.497982
10	17841.43729	10402.26714	10402.26714	10402.26714	17841.43729

Where ADM is Adomian Decomposition Method in [1] , VIM is Variation Iteration Method in [2], HPM(1) is standard Homtopy Perturbation Method and HPM(2) is a new technique using Homotopy Perturbation Method.

Example4[13,18]**:** Consider the following one-dimensional fractional heat-like problem

$$\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}=0,\quad 0<x<1, 0<\alpha<1, t>0, \qquad (4.30)$$

subject to the initial condition

$$u(x,0)=x^2,\ \ 0<x<1, \qquad (4.31)$$

and the boundary conditions

$$u(0,t)=0\,,\quad u(1,t)=\mathrm{E}_\alpha(\mathrm{t}^\alpha)\,,\quad t>0, \qquad (4.32)$$

the exact solution is $u(x,t)=x^2E_\alpha(t^\alpha)$, (4.33)

where,

$$\mathrm{E}_\alpha(\mathrm{t}^\alpha)=\lim_{m\to\infty}\sum_{k=0}^{m}\frac{t^{k\alpha}}{\Gamma(1+k\alpha)}\,. \qquad (4.34)$$

By applying a new approximations u_n^* in Eq. (3.6) we have

$$u_n^*(x,t)=u_n(x,t)+(1-x^2)[0-u_n(0,t)]+x^2[\mathrm{E}_\alpha(\mathrm{t}^\alpha)-u_n(1,t)], \qquad (4.35)$$

where $n=0,1,2,\ldots$. The initial approximation is $u_0(x,t)=x^2$.

To begin with a new initial approximation u_0^* we applying Eq. (4.13) at $n=0$ such as

$$u_0^*(x,t)=x^2\mathrm{E}_\alpha(\mathrm{t}^\alpha)$$
$$=x^2\left(1+\frac{t^\alpha}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\cdots\right). \qquad (4.36)$$

we construct the following homotopy

$$(1-p)\left(\frac{\partial^{\alpha}u}{\partial t^{\alpha}}-\frac{\partial^{\alpha}u_0}{\partial t^{\alpha}}\right)+p\left(\frac{\partial^{\alpha}u}{\partial t^{\alpha}}-\frac{1}{2}x^2\frac{\partial^2u}{\partial x^2}\right)=0, \qquad (4.37)$$

where the initial approximation $u_0(x,0)=x^2$,

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}+p\left(-\frac{1}{2}x^2\frac{\partial^2u}{\partial x^2}\right)=0, \qquad (4.38)$$

we consider u as

$$u=u_0+pu_1+p^2u_3+\cdots \qquad (4.39)$$

substituting (4.39) in (4.38) and equating the coefficients of powers p, we get following set of differential equations with solving the systems accordingly, we obtain,

$$p^0:\frac{\partial^{\alpha}u}{\partial t^{\alpha}}=0, by\ Eq.(4.31) and\ Eq.(1.4), we\ have\ u_0(x,t)=x^2,$$

$$\begin{aligned}p^1:\frac{\partial^{\alpha}u_1}{\partial t^{\alpha}}=\frac{1}{2}x^2\frac{\partial^2u_0^*}{\partial x^2}\Rightarrow u_1 &=\frac{x^2}{2\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}\,\frac{\partial^2(x^2E_{\alpha}(\xi^{\alpha})}{\partial x^2}d\xi\\ &=\frac{x^2}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}E_{\alpha}(\xi^{\alpha})d\xi\\ &=\frac{x^2}{\Gamma(\alpha)}\int_0^t((t-\xi)^{\alpha-1}(1+\frac{\xi^{\alpha}}{\Gamma(1+\alpha)}+\frac{\xi^{2\alpha}}{\Gamma(1+2\alpha)}+\cdots)d\xi,\\ &=\frac{x^2}{\Gamma(\alpha)}(\mathcal{B}(\alpha,1)t^{\alpha}+\frac{\mathcal{B}(\alpha,\alpha+1)t^{2\alpha}}{\Gamma(1+\alpha)}+\frac{\mathcal{B}(\alpha,2\alpha+1)t^{3\alpha}}{\Gamma(2\alpha+1)}+\cdots),\\ &=x^2\left(\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\frac{t^{3\alpha}}{\Gamma(1+3\alpha)}+\cdots\right),\end{aligned}$$

where $\mathcal{B}$is beta function.
Through, the solution of two steps iteration and by setting p =1 in equation.(4.39), the solution of (4.30) can be obtain as

$$u(x,t)=u_0(x,t)+u_1(x,t),$$

$$u(x,t)=x^2\left(1+\frac{t^{\alpha}}{\Gamma(1+\alpha)}+\frac{t^{2\alpha}}{\Gamma(1+2\alpha)}+\cdots\right),$$

$$u(x,t)=x^2E_{\alpha}(t^{\alpha}), \qquad (4.40)$$

which yields an exact solution of Eq.(4.30). it is the same result which is writing by [13] where they are using initial conditions only.

***Example* 5[13,18]:** Consider the one-dimensional fractional wave-like equation

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}}-\frac{1}{2}x^2\frac{\partial^2u}{\partial x^2}=0,\quad 0<x<1, 1<\alpha<2, t>0, \qquad (4.41)$$

subject to the initial conditions

$$u(x,0)=x,\quad \frac{\partial u(x,0)}{\partial t}=x^2,\quad 0<x<1, \qquad (4.42)$$

and the boundary conditions

$$u(0,t)=0,\quad \mathrm{u(1,t)=1+sinht}, \qquad (4.43)$$

the exact solution is$u(x,t)=x+x^2tE_{\alpha}(t^{\alpha})$, (4.44)

by applying a new approximations u_n^* in Eq. (3.6) we obtain

$$u_n^*(x,t)=$$
$$u_n(x,t)+(1-x^2)[0-u_n(0,t)]+x^2[1+\mathrm{sinht}-u_n(1,t)], \qquad (4.45)$$

where $n=0,1,2,\dots$. The initial approximation is $u_0(x,0)=x$.
To begin with a new initial approximation u_0^* we applying Eq. (4.22) at $n=0$ such as

$$\begin{aligned}u_0^*(x,t)&=x+x^2\mathrm{sinht}\\ &=x+x^2t\left(1+\frac{t^3}{\Gamma(4)}+\frac{t^5}{\Gamma(6)}+\cdots\right). \qquad (4.46)\end{aligned}$$

we construct the following homotopy

$$(1-p)\left(\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{\partial^\alpha u_0}{\partial t^\alpha}\right)+p\left(\frac{\partial^\alpha u}{\partial t^\alpha}-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\right)=0 \tag{4.47}$$

where the initial approximation $u_0(x,0)=x$,

$$\frac{\partial^\alpha u}{\partial t^\alpha}+p\left(-\frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}\right)=0, \tag{4.48}$$

we consider u as

$$u=u_0+pu_1+p^2u_3+\cdots. \tag{4.49}$$

Substituting (4.49) in (4.48) and equating the coefficients of powers p, we get following set of differential equations with solving the systems accordingly, we obtain,

$$p^0{:}\,\frac{\partial^\alpha u_0}{\partial t^\alpha}=0, by\ Eq.(4.42), Eq.(1.4) and\ Eq.(1.5), we\ have\ u_0(x,t)=x+x^2t,$$

$$\begin{aligned}p^1{:}\,\frac{\partial^\alpha u_1}{\partial t^\alpha}=\frac{1}{2}x^2\frac{\partial^2 u_0^*}{\partial x^2}\Rightarrow u_1&=\frac{x^2}{2\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}\frac{\partial^2(x+x^2 sinh\xi)}{\partial x^2}d\xi\\&=\frac{x^2}{\Gamma(\alpha)}\int_0^t(t-\xi)^{\alpha-1}\sinh\xi\, d\xi,\\&=\frac{x^2}{\Gamma(\alpha)}\int_0^t((t-\xi)^{\alpha-1}(1+\frac{\xi^3}{\Gamma(4)}+\frac{\xi^5}{\Gamma(6)}+\cdots)d\xi,\\&=\frac{x^2}{\Gamma(\alpha)}(\mathcal{B}(\alpha,2)t^{\alpha+1}+\frac{\mathcal{B}(\alpha,4)t^{\alpha+3}}{\Gamma(4)}+\frac{\mathcal{B}(\alpha,6)t^{\alpha+5}}{\Gamma(6)}+\cdots),\\&=x^2\left(\frac{t^{\alpha+1}}{\Gamma(2+\alpha)}+\frac{t^{\alpha+3}}{\Gamma(4+\alpha)}+\frac{t^{\alpha+5}}{\Gamma(6+\alpha)}+\cdots\right),\end{aligned} \tag{4.50}$$

through, the solution of two steps iteration and by setting $p=1$ in equation.(4.50), the solution of (4.41) can be obtain as

$$u(x,t)=u_0(x,t)+u_1(x,t),$$

$$u(x,t)=x+x^2\left(t+\frac{t^{\alpha+1}}{\Gamma(2+\alpha)}+\frac{t^{\alpha+3}}{\Gamma(4+\alpha)}+\frac{t^{\alpha+5}}{\Gamma(6+\alpha)}+\cdots\right). \tag{4.51}$$

Let $\alpha=2$, Eq.(4.51) becomes

$$\mathrm{u(x,t)}=\mathrm{x}+\mathrm{x}^2\left(\mathrm{t}+\frac{\mathrm{t}^3}{3!}+\frac{\mathrm{t}^5}{5!}+\cdots\right)=\mathbf{x}+\mathrm{x}^{\mathbf{2}}\mathrm{sinht}. \tag{4.52}$$

Which yields an exact solution of Eq. (4.41) when $\alpha=2$. But, generally the solution (4.51) is not exactly for Eq. (4.41), because it doesn't satisfying of vinery equation. In the other hand the boundary conditions (4.42) is not corresponding to the exact solution (4.43) which were given in [13,18], that's mean the boundary conditions (4.43) are be not true. So, in the following example we present a reliable framework by applying another boundary conditions which can be satisfying by exact solution (4.41).

Table.4. Approximation(analytic)solutions, x=0.8

t	Exact solution	VIM $n = 10, \alpha = 2$	ADM $n = 10, \alpha = 2$	HPM(1) $n = 10, \alpha = 2$	HPM(2) $n = 1, \alpha = 2$
0	0.800000000	0.800000000	0.80000000	0.800000000	0.800000000
1	1.552128764	1.552128764	1.552128764	1.552128764	1.552128764
2	3.121190661	3.121190661	3.121190661	3.121190661	3.121190661
3	7.211439955	7.211439949	7.211439949	7.211439949	7.211439955
4	18.26554701	18.26554701	18.26554701	18.26554701	18.26554701
5	48.29005477	48.29005446	48.29005446	48.29005446	48.29005477
6	129.8964207	129.8964000	129.8964000	129.8964000	129.8964207
7	351.7223189	351.7215819	351.7215819	351.7215819	351.7223189
8	954.7064486	954.6901222	954.6901222	954.6901222	954.7064486
9	2593.786817	2593.533992	2593.533992	2593.533992	2593.786817
10	7049.269037	7046.313933	7046.313933	7046.313933	7049.269037

Where ADM is Adomian Decomposition Method in [1] , VIM is Variation Iteration Method in [2], HPM(1) is standard Homtopy Perturbation Method and HPM(2) is a new technique using Homotopy Perturbation Method.

***Example* 6[13,18]:** Consider the one-dimensional fractional wave-like equation

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{1}{2} x^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, 1 < \alpha \leq 2, t > 0, \tag{4.53}$$

subject to the initial conditions

$$u(x,0) = x, \quad \frac{\partial u(x,0)}{\partial t} = x^2, \quad 0 < x < 1, \tag{4.54}$$

and the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 1 + t\,\mathrm{E}_{\alpha,2}(\mathrm{t}^\alpha), \quad t > 0, \tag{4.55}$$

the exact solution is

$$u(x,t) = x + x^2 t E_\alpha(t^\alpha), \tag{4.56}$$

where

$$\mathrm{E}_{\alpha,2}(\mathrm{t}^\alpha) = \lim_{\mathrm{m}\to\infty} \sum_{\mathrm{k}=0}^{\mathrm{m}} \frac{\mathrm{t}^{\mathrm{k}\alpha}}{\Gamma(2+\mathrm{k}\alpha)}. \tag{4.57}$$

By applying a new approximations u_n^* in Eq. (3.6) we obtain

$$u_n^*(x,t) = u_n(x,t) + (1 - x^2)[0 - u_n(0,t)] + x^2[1 + t\,\mathrm{E}_{\alpha,2}(\mathrm{t}^\alpha) - u_n(1,t)], \tag{4.58}$$

where $n = 0,1,2,\dots$. The initial approximation is $u_0(x,0) = x$.
To begin with a new initial approximation u_0^* we applying Eq. (4.22) at $n = 0$ such as

$$\begin{aligned} u_0^*(x,t) &= x + x^2 t\,\mathrm{E}_{\alpha,2}(\mathrm{t}^\alpha) \\ &= x + x^2 t \left(1 + \frac{t^\alpha}{\Gamma(2+\alpha)} + \frac{t^{2\alpha}}{\Gamma(2+2\alpha)} + \cdots\right). \end{aligned} \tag{4.59}$$

we construct the following homotopy

$$(1-p)\left(\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{\partial^\alpha u_0}{\partial t^\alpha}\right) + p\left(\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{1}{2} x^2 \frac{\partial^2 u}{\partial x^2}\right) = 0, \tag{4.60}$$

where the initial approximation $u_0(x,0) = x$,

$$\frac{\partial^\alpha u}{\partial t^\alpha} + p\left(-\frac{1}{2} x^2 \frac{\partial^2 u}{\partial x^2}\right) = 0, \tag{4.61}$$

we consider u as

$$u = u_0 + p u_1 + p^2 u_3 + \cdots \tag{4.62}$$

substituting (4.62) in (4.61) and equating the coefficients of powers p, we get following set of differential equations with solving the systems accordingly, we obtain,

$$p^0: \frac{\partial^\alpha u_0}{\partial t^\alpha} = 0, by\ Eq.(4.54), Eq.(1.4) and\ Eq.(1.5), we\ have\ \ u_0(x,t) = x + x^2 t,$$

$$\begin{aligned} p^1: \frac{\partial^\alpha u_1}{\partial t^\alpha} = \frac{1}{2} x^2 \frac{\partial^2 u_0^*}{\partial x^2} \Rightarrow u_1 &= \frac{x^2}{2\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} \frac{\partial^2 (x + x^2 t E_\alpha(\xi^\alpha)}{\partial x^2} d\xi \\ &= \frac{x^2}{\Gamma(\alpha)} \int_0^t (t-\xi)^{\alpha-1} \xi E_\alpha(\xi^\alpha) d\xi \\ &= \frac{x^2}{\Gamma(\alpha)} \int_0^t ((t-\xi)^{\alpha-1} (1 + \frac{\xi^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{\xi^{2\alpha+1}}{\Gamma(2+2\alpha)} + \cdots) d\xi, \\ &= \frac{x^2}{\Gamma(\alpha)} (\mathcal{B}(\alpha,2) t^{\alpha+1} + \frac{\mathcal{B}(\alpha,\alpha+2) t^{2\alpha+1}}{\Gamma(2+\alpha)} + \frac{\mathcal{B}(\alpha,2\alpha+2) t^{3\alpha+1}}{\Gamma(2+2\alpha)} + \cdots), \\ &= x^2 \left(\frac{t^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{t^{2\alpha+1}}{\Gamma(2+2\alpha)} + \frac{t^{3\alpha+1}}{\Gamma(2+3\alpha)} + \cdots \right), \qquad (4.63) \end{aligned}$$

where $\mathcal{B}$is beta function.

Through, the solution of two steps iteration and by setting $p = 1$ in equation.(4.62), the solution of (4.53) can be obtain as

$$u(x,t) = u_0(x,t) + u_1(x,t),$$

$$u(x,t) = x + x^2 t \left(1 + \frac{t^\alpha}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots \right),$$

$$u(x,t) = x + x^2 t E_{\alpha,2}(t^\alpha),$$

which yields an exact solution of Eq.(4.53), it is the same result which is writing by [13] where they are using initial conditions only.

5. Conclusions

This paper develops the treatment of construct a new initial successive solutions u_n^* by mixed initial and boundary conditions together which explained in formula (3.6) and used it to find successive approximations u_n of the solution u by applying homotopy perturbation method to solve initial boundary value problems for one dimensional heat-like and wave-like partial differential equations (ordinary or fractional). Some examples are given in this paper to illustrate the effectiveness and convenience of a new technique. It is important and obvious to show that the exact solutions have found directly from a first iteration of these examples by applying a new technique which is determined in this paper, but if used initial conditions only [13,18] or applied formula of Eq. (3.4) [1,2] we will have exact solution by calculating infinite successive solutions u_n which closed form by Eq. (2.9). In the other hand we note that the first a new initial approximation u_0^* are appearing in the same exact solution.

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معالجة مطورة للمسائل ذات القيم الابتدائية والحدودية الخاصة بالمعادلات المثيلة للحرارية والموجية ذات البعد الواحد باستخدام طريقة الاضطراب الهوموتوبي

المستخلص:

في هذا البحث، طبقنا معالجة مطورة لمسائل ذات قيم ابتدائية وحدودية الخاصة بالمعادلات المثيلة للحرارية والموجية ذات البعد الواحد (تفاضلية اعتيادية أو كسرية) وذلك بخلط الشروط الابتدائية والحدودية معا بأسلوب معين لغرض الحصول على حل ابتدائي جديد عند كل خطوة تكرارية باستخدام طريقة الاضطراب الهوموتوبي (HPM) ، أن تكوين متتابعة الحلول الابتدائية بالأسلوب الجديد يعطي حلا دقيق في أول خطوة تكرارية.