Solving System of Fuzzy Linear Differential Equations With Fuzzy Initial Values

حل نظام خطي من المعادلات التفاضلية الضبابية مع القيم الضبابية الأولية

Assis. Lect. Athera Nema Kathem
Department of Mathematics. College of Education. University of Kufa

Abstract :

In this paper, we consider a system of linear fuzzy differential equations (F.D.Es) with fuzzy initial values. We discuss the solution of the system

$\dot{x}(t) = Ax(t) + f(t, g)\ ; t > 0$

$x(0) = x_0$,

where A is a real n × n matrix, the initial condition x_0 and the constant g are vectors of n fuzzy numbers and the function f(t,g) is described by multiplication the variable t with the vector of fuzzy numbers g. In addition , we generalized the solution for solving nth order linear F.D.Es. with fuzzy initial values .

المستخلص:

في هذا البحث أخذنا بنظر ألاعتبار نظام خطي من المعادلات التفاضلية الضبابية مع القيم الضبابية الأولية , فناقشنا حل النظام $\dot{x}(t) = Ax(t) + f(t, g)\ ; t > 0 : x(0) = x_0$ حيث أن A مصفوفة حقيقية سعة n × n , الشرط الابتدائي X_0 والثابت g متجهان من n من الإعداد الضبابية , الدالة f(t,g) وصفت بحاصل ضرب المتغير t مع متجه ألاعداد الضبابية g .

1 .Introduction

The topic of F.D.Es. has been rapidly growing in recent years. The concept of fuzzy derivative was first introduced by Chang and Zadeh [1], it was followed up by Dobbis and Prade[2] who used the extension principle in their approach . Other methods have been discussed Puri and Ralesuc [3]. In addition, one of the important application of fuzzy numbers is treating systems of linear fuzzy differential equations [4] and [5] .

In section 2, we preview some definitions on fuzzy numbers and linear fuzzy systems.

In section 3, we introduced a formula for solving the system of linear F.D.Es. with fuzzy initial values

$$\dot{x}(t) = Ax(t) + f(t, g)\ ; t > 0$$
$$x(0) = x_0 \ .$$

In section 4, we generalize this method in order to solve the n-th order linear fuzzy differential equations with fuzzy initial values .

2. Preliminaries

The Prime information of an arbitrary fuzzy numbers is given in [6] and we give some necessary definitions which will be used in this paper .

Definition 2-1 [5] :A fuzzy u is a pair ($\underline{u}$, $\bar{u}$) of functions $\underline{u}(r)$, $\bar{u}(r)$, $0 \le r \le 1$ which satisfy the following requirements .

1 . $\underline{u}$ is a bounded left continuous non-decreasing function over [0 , 1] .
2 . $\bar{u}$ is a bounded left continuous non-increasing function over [0 , 1] .
3 . $\underline{u}(r) \le \bar{u}(r)$, $0 \le r \le 1$.

The set of all these fuzzy numbers is denoted by E which is a complete metric space Hausdorff distance .

Remark 2-1 [6]

A crisp number u is simply represented by $\underline{u}(r) = \bar{u}(r)$, $0 \le r \le 1$.

For two arbitrary fuzzy numbers x= ($\underline{x}(r)$, $\bar{x}(r)$) , y= ($\underline{y}(r)$, $\bar{y}(r)$) and real number k , we may define the addition and the scalar multiplication of fuzzy numbers by using the extension principle as

i . $x = y$ if and only if $\underline{x}(r) = \underline{y}(r)$ and $\overline{x}(r) = \overline{y}(r)$,

ii . $x + y = (\underline{x}(r) + \underline{y}(r) \ , \ \overline{x}(r) + \overline{y}(r))$,

iii . $k\,x = \begin{cases} (k\underline{u}, k\overline{u}), k \geq 0, \\ (k\overline{u}, k\underline{u}), k < 0, \end{cases}$

Definition 2-2 [5] : The $n \times n$ linear system

$$\begin{aligned} &a_{11}x_1 + \ldots + a_{1n}x_n = y_1 \ , \\ &a_{21}x_1 + \ldots + a_{2n}x_n = y_2 \ , \\ &\ \vdots \\ &a_{n1}x_1 + \ldots + a_{nn}x_n = y_n \ , \end{aligned} \qquad \ldots \quad (1)$$

where the given matrix of coefficients $A =(a_{ij})$, $1 \leq i , j \leq n$, is a real $n \times n$ matrix , the given $y_i \in E$, $1 \leq i \leq n$, with the unknown $x_j \in E$, $1 \leq j \leq n$ is called a linear fuzzy system (L.F.S.) .

Definition 2-3 [7] :A fuzzy number vector $(x_1 ,x_2 , \ldots, x_n)^t$ given by

$x_j= (\underline{x}_j(r) , \overline{x}_j(r))$, $1 \leq j \leq n$, $0 \leq r \leq 1$,

is called a solution of the linear fuzzy system (1) if

$$\begin{aligned} \underline{\sum_{j=1}^{n} a_{ij}\,x_j} = \sum_{j=1}^{n} \underline{a_{ij}\,x_j} = \underline{y_i} \\ \overline{\sum_{j=1}^{n} a_{ij}\,x_j} = \sum_{j=1}^{n} \overline{a_{ij}\,x_j} = \overline{y_i} \end{aligned} \qquad \ldots \quad (2)$$

If , for a particular i , $a_{ij} > 0$ for all j , we simply get

$$\sum_{j=1}^{n} a_{ij}\,\underline{x_j} = \underline{y_i} \qquad , \qquad \sum_{j=1}^{n} a_{ij}\,\overline{x_j} = \overline{y_i}$$

from (2) we have two crisp $n \times n$ linear systems for all i that they can be extended to a $2n \times 2n$ crisp linear system as follows

$MX = Y$, $\qquad \ldots \quad (3)$

or

$$\begin{pmatrix} M_1 & M_2 \\ M_3 & M_4 \end{pmatrix} \begin{pmatrix} \underline{x} \\ \overline{x} \end{pmatrix} = \begin{pmatrix} \underline{Y} \\ \overline{Y} \end{pmatrix} \ ,$$

where $(\underline{X} , \overline{X})^t = (\underline{x} , \ldots, \underline{x} , \overline{x}, \ldots , \overline{x})^t$ and s_{ij} are determined as

$$\left.\begin{aligned} a_{ij} \geq 0 \ &\Rightarrow \ m_{ij} = m_{i+n,j+n} = a_{ij} \ , \\ a_{ij} \leq 0 \ &\Rightarrow \ m_{i+n,j} = m_{i,j+n} = a_{ij} \ , \end{aligned}\right\} \qquad \ldots \quad (4)$$

and any s_{ij} which is not determined is zero such that

$A = M_1 + M_2$.

3 . System of linear fuzzy differential equations with fuzzy initial value.

We are discussed the solution of the system of F.D.Es. with fuzzy initial values

$$\dot{x}(t;r) = Ax(t;r) + f\big(t;g(r)\big) \ ; t > 0 \qquad \ldots \quad (5)$$
$$x(0;r) = x_0 \ ,$$

where $\dot{x} = \frac{dx}{dt}$, A is a real $n \times n$ matrix , the initial condition x_0 and the function $g(r)$ are vectors of n fuzzy numbers and the function $f(t,g(r))$ is described by multiplication the variable t with the vector of fuzzy numbers $g(r)$. Let I be a real interval , a mapping $x : I \to E$ is called a fuzzy process, the derivative $\dot{x}(t)$ of a fuzzy process $x(t)$ is defined by

$[\dot{x}(t;r)] = [\underline{\dot{x}}(t;r), \overline{\dot{x}}(t;r)]$, $0 \le r \le 1$,
provided that is equation defines a fuzzy number [1].

Consider the system of linear fuzzy differential equations with fuzzy initial conditions (5) , and transform its n x n coefficients matrix A into $2n \times 2n$ matrix as in equation (4) . Define matrix

$M = (m_{ij})$, $1 \le i , j \le n$

$a_{ij} \ge 0 \Rightarrow m_{ij} = m_{i+n,j+n} = a_{ij}$,

$a_{ij} \le 0 \Rightarrow m_{i+n,j} = m_{i,j+n} = a_{ij}$,

while all the remaining m_{ij} is taken zero . Using matrix nation , we get

$$\dot{X}(t) = MX(t) + F(t, G(r)), t > 0 \quad \ldots(6)$$
$$X(0) = X_0$$

where $M = (m_{ij})$ $0 \le i , j \le 2n$ and

$$X(t;r) = \begin{bmatrix} \underline{x}_1(t;r) \\ \vdots \\ \underline{x}_n(t;r) \\ \overline{x}_1(t;r) \\ \vdots \\ \overline{x}_n(t;r) \end{bmatrix} , \quad \dot{X}(t;r) = \begin{bmatrix} \underline{\dot{x}}_1(t;r) \\ \vdots \\ \underline{\dot{x}}_n(t;r) \\ \overline{\dot{x}}_1(t;r) \\ \vdots \\ \overline{\dot{x}}_n(t;r) \end{bmatrix} , \quad \ldots (7)$$

$$X_0 = \begin{bmatrix} \underline{x}_{01} \\ \vdots \\ \underline{x}_{0n} \\ \overline{x}_{01} \\ \vdots \\ \overline{x}_{0n} \end{bmatrix} , \quad G(r) = \begin{bmatrix} \underline{g}_1(r) \\ \vdots \\ \underline{g}_n(r) \\ \overline{g}_1(r) \\ \vdots \\ \overline{g}_n(r) \end{bmatrix} , \quad F(t;G(r)) = \begin{bmatrix} t\,\underline{g}_1(r) \\ \vdots \\ t\,\underline{g}_n(r) \\ t\,\overline{g}_1(r) \\ \vdots \\ t\,\overline{g}_n(r) \end{bmatrix} , \quad \ldots (8)$$

Assuming that M is a non singular . The solution to equation (6) is then given by

$$X(t;r) = -(t\,M^{-1}F(t;G(r)) + M^{-2}F(t;G(r)) + e^{tM}(X_0 + M^{-2}F(t;G(r))) \quad \ldots (9)$$

and the problem is therefore to calculate the exponential of matrix M . Then proceed to calculate the formula series of e^{tM} :

$$e^{tM} = I + tM + \frac{t^2}{2!}M^2 + \frac{t^3}{3!}M^3 + \cdots \frac{t^n}{n!}M^n \quad \ldots (10)$$

4 . System of the linear F.D.Es. of order n with fuzzy initial values .

In this section , we introduced the general formula of solution for nth order linear F.D.E.

$$\left(\frac{d^n}{dt^n} + a_1\frac{d^{n-1}}{dt^{n-1}} + \cdots + a_n\right)y = \omega\left(t;\mu(r)\right) \quad ; \ t > 0 \quad , \quad \ldots \quad (11)$$

with the initial values

$$y(0) = \varphi_1 \ , \dot{y}(0) = \varphi_2, \ldots, y^{n-1}(0) = \varphi_n \quad , \quad \ldots \quad (12)$$

where

$a_1, \ldots , a_n \in R$, μ , ω , φ_i , $\frac{d^n y}{dt^n}$ and $\frac{d^{i-1}y}{dt^{i-1}} \in E$, $i = 1 , \ldots , n$.

Also, we can used the formula (9) to solve the equation (11) with fuzzy initial conditions (12) after transforming to a system of n linear fuzzy differential equations (5) by the assumptions [4]

$$x_1(t) = y(t) , x_2(t) = \frac{dy}{dt} , \ldots , x_n(t) = \frac{d^{n-1}y}{dt^{n-1}} \quad ,$$

So these assumption gives the system of linear fuzzy differential equations

$$\dot{x}_1(t) = x_2(t)$$
$$\dot{x}_2(t) = x_3(t)$$
$$\vdots$$
$$\dot{x}_n(t) = \sum_{i=1}^{n} a_i x_{n+1-i}(t) + \omega(t;\mu(r)) \quad ,$$

with fuzzy initial conditions

$$x_1(0) = \varphi_1 \, , \; x_2(0) = \varphi_2 \, , \ldots , x_n(0) = \varphi_n \quad . \qquad \ldots \quad (13)$$

Therefore, any n th order initial values problem of the form (11) with the initial vales(12) can be transformed to the system of fuzzy differential equations (5) and apply this method to find the general formula (9) of approximate solution when

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_n & -a_{n-1} & a_{n-2} & \cdots & -a_2 & -a_1 \end{bmatrix}, f(t;g(r)) = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \omega(t;\mu(r)) \end{bmatrix},$$

with the fuzzy conditions given in (13) .

Example (1)

we introduce a simple example in special case for homogenous linear fuzzy differential equations

$$\dot{x}(t) = \begin{bmatrix} 2 & -3 \\ 0 & 1 \end{bmatrix} x(t) \quad , \qquad \ldots \quad (14)$$

and defining the initial values to be x_{01} = about 2 and x_{02} = about 1 which can be done by setting for example

$x_{01} = x_1(0;r) = (r , 2-r)$

$x_{02} = x_2(0;r) = (3+r , 5-r)$

since f(t , g(r)) = 0 ,

by equations (4),(6),(7) and (8) , we get

$$\begin{bmatrix} \dot{\underline{x}}_1 \\ \dot{\underline{x}}_2 \\ \dot{\overline{x}}_1 \\ \dot{\overline{x}}_2 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 2 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \overline{x}_1 \\ \overline{x}_2 \end{bmatrix} \quad , X_0 = \begin{bmatrix} r \\ 3+r \\ 2-r \\ 5-r \end{bmatrix}$$

we can obtain the approximate solution by equation (9) and (10) ,for t = (0.1) the approximate solution given in table 1 and 2 :

N	$\underline{x}_1(0.1;r)$	$\overline{x}_1(0.1;r)$
1	1.2 r	2.4 – 1.2 r
2	1.22 r	2.44 – 1.22 r
3	1.2213 r	2.4426 – 1.2213 r
4	1.221400 r	2.442800 – 1.221400 r
5	1.22140266 r	2.44280533 – 1.22140266 r
6	1.221402755 r	2.442805511 - 1.221402755r
7	1.2214027580 r	2.4428055161 - 1.2214027580 r

Table 1 .The approximate solution of $x_1(0.1;r)$

N	$\underline{x}_2(0.1;r)$	$\overline{x}_2(0.1;r)$
1	2.7 + 1.4 r	5.5 – 1.4 r
2	2.625 + 1.450 r	5.525 – 1.450 r
3	2.61850 + 1.45366 r	5.52583 – 1.45366 r
4	2.6181375 + 1.4538583 r	5.5258541 – 1.4538583 r
5	2.61812225 + 1.45386616 r	5.52585458 – 1.45386616 r
6	2.618121712 +1.453866430 r	5.525854590 - 1.453866430 r
7	2.6181217141 + 1.4538664381 r	5.5258545903 - 1.4538664381 r

Table 2 .The approximate solution of $x_2(0.1;r)$

At t = 0.1 , one can find the exact solution of the system (14) as

$x_1(0.1;r)$ =(1.221402758 r , 2.442805516 – 1.221402758 r) ,

$x_2(0.1;r)$ = (2.618121714 + 1.453866438 r , 5.52585459 – 1.453866438 r) .

Example (2)

In general case we present the non - homogenous linear fuzzy differential equations by defining $A=\begin{bmatrix}1 & -1\\ 0 & 4\end{bmatrix}$

and defining a function f(t ,g(r)) from described g(r) by

$g_1(r) = (r , 2-r)$ and $g_2(r) = (2+r , 4-r)$,therefore ,

$$f(t, g(r)) = t\begin{bmatrix}(r & , & 2-r)\\ (2+r, & & 4-r)\end{bmatrix} ,$$

so f(t, g(r)) when t > 0 is a fuzzy vector [7] . We have

$$\dot{x}(t) = \begin{bmatrix}1 & -1\\ 0 & 4\end{bmatrix}x(t) + t\begin{bmatrix}(r & , & 2-r)\\ (2+r, & & 4-r)\end{bmatrix} , \qquad \dots \quad (15)$$

with fuzzy initial conditions

$x_{01} = x_1(0;r) = (1+r , 3-r)$, $x_{02} = x_2(0;r) = (r , 2-r)$

by equations (4),(6),(7) and (8) , we get

$$\begin{bmatrix}\dot{\underline{x}}_1\\ \dot{\underline{x}}_2\\ \dot{\overline{x}}_1\\ \dot{\overline{x}}_2\end{bmatrix} = \begin{bmatrix}1 & 0 & 0 & -1\\ 0 & 4 & 0 & 0\\ 0 & -1 & 1 & 0\\ 0 & 0 & 0 & 4\end{bmatrix}\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\\ \overline{x}_1\\ \overline{x}_2\end{bmatrix} + \begin{bmatrix}rt\\ (2+r)t\\ (2-r)t\\ (4-r)t\end{bmatrix} , X_0 = \begin{bmatrix}1+r\\ r\\ 3-r\\ 2-r\end{bmatrix}$$

we can obtain the approximate solution by equation (9) and (10) as follows :

$$\begin{bmatrix}\underline{x}_1\\ \underline{x}_2\\ \overline{x}_1\\ \overline{x}_2\end{bmatrix} = \begin{bmatrix}-1.3500-0.7625r\\ -1.7500-0.0875r\\ -2.8750+0.7625r\\ -0.3500+0.0875r\end{bmatrix} + K_n\begin{bmatrix}2.2500+1.6875r\\ 0.1250+1.0625r\\ 5.6250-1.6875r\\ 2.2500-1.0625r\end{bmatrix}$$

where $K_n = I + tM + \frac{t^2}{2!}M^2 + \frac{t^3}{3!}M^3 + \cdots \frac{t^n}{n!}M^n$

For t = (0.1) the approximate solution shown in table 1 and 2 :

N	$\underline{x}_1(0.1;r)$	$\overline{x}_1(0.1;r)$
1	0.9 + 1.2 r	3.3 – 1.2 r
2	0.855 + 1.235 r	3.325 – 1.235 r
3	0.8475 + 1.239 r	3.3255 – 1.239 r
4	0.8467125 + 1.2393833 r	3.32547916 – 1.2393833 r
5	0.84664875 + 1.23941333 r	3.32547608 – 1.23941366 r
6	0.8466444875 + 1.2394156833 r	3.3254758541 - 1.2394156833 r
7	0.8466442437 + 1.2394157984 r	3.3254758407 - 1.2394157984 r

Table 1 .The approximate solution of $x_1(0.1;r)$

N	$\underline{x}_2(0.1;r)$	$\overline{x}_2(0.1;r)$
1	0 + 1.4 r	2.8 – 1.4 r
2	0.010 + 1.485 r	2. 980 – 1.485 r
3	0.0113 + 1.4963 r	3.0040 – 1.4963 r
4	0.0114666 + 1.4974666 r	3.0064000 – 1.4974666 r
5	0.011477333 + 1.497557333 r	3.006592000 – 1.497557333 r
6	0.0114780444 + 1.4975633777 r	3.0066048000 - 1.4975633777 r
7	0.0114780850 + 1.4975637231 r	3.0066055314 - 1.4975637231 r

Table 2 .The approximate solution of $x_2(0.1;r)$

Also, for $t = 0.1$, we can find the exact solution of the system (15) as

$x_1(0.1;r) =(0.846644231+1.239465531r , 3.325475841 – 1.239415805 r)$,

$x_2(0.1;r) = (0.011478087 + 1.497563742 r , 3.006605571 – 1.497563742 r)$.

References

[1] S. L. Chang and L. A. Zadeh, On fuzzy mapping and control, IEEE Trans. Systems Man Cybemet. 2, (1972) 30-34 .

[2] D. Dubois and H. Prade, Towards fuzzy differential calculus :Part 3, differentiation , Fuzzy Sets and Systems 8, (1982) 225-233 .

[3] M. L. Puri and D. A. Ralescu, Differentials of fuzzy functions, J. Math. Anal. Apple. 91 , (1983) 552-558 .

[4] A. K. Haydar, A. N. Kadhim and R. A. Kuder, A method for solving system of non-homogenous linear fuzzy differential equations, Journal of Babylon University. No. 5 ,2012 .

[5] M. Otadi, S. Abbasbandy and M. Masleh, System of linear fuzzy differential equations, First Joint Congress on Fuzzy and Intelligent Systems, Ferdowsi University of Mashah, Iran, 2007 .

[6] M. Friedman, Ma Ming and A. Kandel, A new fuzzy arithmetic, Fuzzy Sets and Systems 108, (1999) 83-90 .

[7] M. Friedman, Ma Ming and A. Kandel, Duality in fuzzy linear systems, Fuzzy Sets and Systems 109 , (2000) 55-58 .