Diameter and width via Rotating Calipers of a Set of Two-dimensional Convex Hull Vertices using Graham scan algorithm

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Abstract

A convex hull is an important shape in graphic geometry that can be used to construct many other geometric structures.In this paper we compute the Convex hull polygon by eliminating all the interior points and bounding box the exterior points using Graham scan algorithm then we compute the Diameter and width of aconvex hull polygon using a powerful, easy and well-designedmethod that can solve several computational geometric difficultiesprofessionallyin practice called Rotating Caliper method. The proposed idea of a well-known rotating caliper algorithm includesrotate pairs ofparallel straight lines called antipodal pairs around a given polygon's vertices, the first one located inmaximum Y-coordinate and the second one is in minimum Y-coordinateof Convex hull vertices. If wecontinuously sweep the tangent of the antipodal pairs through 360 degrees,then the width and diameter of the convex hull polygon are simply detected byminimum and maximum distance between the antipodal pairs during thissweep process. The experimental results revealthat The rotating calipers algorithm is fast ,powerful, easy and general tool for solving geometric problems.

Keywords: Rotating Calipers; Computational Geometric; Graham Convex Hull.

المستخلص

التل المحدب هوشكل مهم في الرسم الهندسي يمكن استخدامه لبناء العديد من الاشكال الهندسية الأخرى. في هذه البحث تم حساب متعدد الاضلاع المحدب من خلال التخلص من جميع النقاط الداخلية واحاطة النقاط الخارجية فقط باستخدام مفهوم bounding box استخدام خوارزمية مسح غراهام ثم حساب القطروالعرض لمتعدد الاضلاع المحدب. تعد خوارزمية الفرجار الدواراحد الخوارزميات القوية ، السهلة و المصممة بشكل جيد لحل العديد من المشاكل الهندسية الحسابية بكفاءة في الجوانب العملية. فكرة الخوارزمية المقترحة والمعروفة بالفرجارالدواريشمل تدوير أزواج من الخطوط المستقيمة المتوازية تسمى أزواج تقابلية حول القمم لمتعدد الاضلاع المعطى ، الخط الأول يقع في أقصىY- تنسيق ثنائي الابعاد والاخريقع في الحد الأدنىY- تنسيق ثنائي الابعاد للتل المحدب. في كل مرة نحصل فيها على أزواج تقابلية من خلال التدوير بزوايا حتى الوصول الى زاوية 360 درجة ، يتم الكشف ببساطة عن العرض والقطر لمتعدد الأضلاع المحدب من خلال حساب المسافة بينا لحد الأدنى والحد الأقصى للأزواج التقابلية خلال عملية التدوير.النتائج التجريبية تكشف عن أن الخوارزمية لفرجار الدورية هي أداة سريعة وقوية وسهلة وعامة من أجل حل معظم المشاكل الهندسي.

1. Introduction

Computational Geometry is the study of geometric problems and thealgorithms to solve them. Problems range from the direct to the verymultifaceted in their details [2]. Let P= (P0, P1, ... , Pn) be a sequence of two dimension points orderin Cartesian coordinate plane, we present a familiar algorithm for solving many problems concerning about building a convex hull polygon from a given two-Dimension set points in plane. Then, a more instinctive of rotating calipersalgorithm used to computing the width and diameter by computing minimum andmaximum distance between two parallel lines named antipodal pair around the convex polygon vertices or boundary[5].Calculationthe width or diameter of convex polygon has many applications in collisionpreventiondifficulties and in approximating polygonal shape curves. In this researchalgorithm to find the diameteror width of a convex hull polygon will be offered,depended on Shamos' "rotating calipers" algorithm for calculating the diameter or width. The algorithms run in O(n) time and O(n) space complexity[1]. Figure (1) shows theconcept of Rotating calipers.In the current paper, we first consider the convex hull of the setof points using Graham scan algorithm. Second, scan the hull and find antipodalpairs. Then, Compute the distances between antipodal pairs, Report thelargest and smallest such distance.Finally, the conclusions and references.

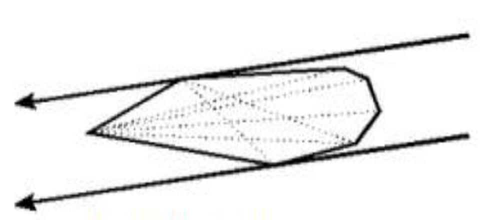


Figure (1) Rotating calipers

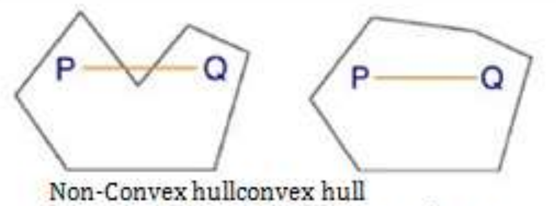
2. Related Work

Related work on finding a maximum and minimum distance of convexhull polygon. The Research done in 1978by M.I.Shamos started the field ofComputational Geometry within theoretical Computer Science, his Ph.D.thesis carrying the same name. In his work, he discusses the problem of findingthe diameter of a convex polygon and gives a fast and well-organized algorithm as asolution. This algorithm is the basis for the Rotating Calipers procedure[1].The term "rotating calipers" later coined in 1983 by G.T. Toussaint tosolve Computationalgeometry problems by computing distances such as finding diameter of a convex hullpolygon or computing Enclosing bounding box for instance the minimum area ofenclosing rectangle [4]. Another piece of work in 1999 by H. Pirzadeh, he use Rotating Calipers as a method to solve a number of problems in the field of ComputationalGeometry. The resulting algorithms are often optimal in their time and space complexity,and always efficient and powerful to implement [3]. Since then, the Rotating Calipers paradigm was also been generalized toapplications of three dimension points [6], although we limit the scope of this paper to solve some ofcomputational geometric problems in two dimensional problems.

3. Computing a 2-D Convex Hull: Graham's Algorithm

3-1 Graham scan algorithm

Computing a convex hull (or just "hull") is one of the first sophisticatedcomputational geometry algorithms, and there are many variations of it [8]. The mostpublic form of this algorithm involves determining the smallest convex set(called the "convex hull") containing a discrete set of two dimensional points in the plane [7]. Aconvex hullCH means that a subset of space is called convex if and only if for everypair of points P and Q in the set of CH, the line segment from P point to Q point is enclosedcompletely in the set of CH [9], as portrayed on the figure(2) below.



Non-Convex hullconvex hull
Figure(2) convex and non-convex polygon

The most popular algorithms for computing convex hullCH are the "Graham scan" algorithm of a set P of n points in the plane [11]. The algorithm isfascinating for a number ofreasons. First of all, it is simple to compute and is veryintuitive. Moreover, for a Data Structuresclass it is quite convincing. Thisalgorithm describes bellow [9].

Graham scan algorithm

Input:A set P= (P0, P1, ... , Pn) of two- dimension points in the plane.

Output: Convex Hull CH(S) contains S vertices.

Start

Step1: Find an extreme point p0 of P that is a vertex of CH and denote it as theanchor point,
We can for example, pick as our anchor point the point in P with minimum y-coordinate value (in screen coordinate p0 hasmaximum y-coordinate) .

Step2: Translate the anchor point of P and all the other point, so the extreme point is at the origin bysubtract each point of P from anchor or extreme point [11].

Step3: Find the angle between the line connecting P0 to each of the points and the positive X- axis using the following steps[17].

- Find differences between the X-axis of two points. deltaX =Px - P0x
- Find differences between the Y-axis of two points. deltaY =Py - P0y
- if (deltaX = 0)and (deltaY = 0)then Angle = 0 else
- Compute angle in radian degree.

angle= ArcTan2(deltaY,deltaX)*57.295779513082 ------------- (1)
if (angle < 0)then add 360 to angle.

Step4: Sort the points in order of increasing angle about anchor point using Mergesort algorithm. The Resulting sorted list of points order in counterclockwise (CCW)with respect to the extreme point [12].

Step5: If two points have the same value of angle, delete the point with the smaller amplitude or distance to anchor point using Euclidean distance measure[10], as in equation(1).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \text{ ------------- (2)}$$

Step6: Starting from the anchor point p0and in CCW manner, labelthe points (P0, P1, P2, ...) It looks like a fan with anchor at the point P0, as in figure (3).

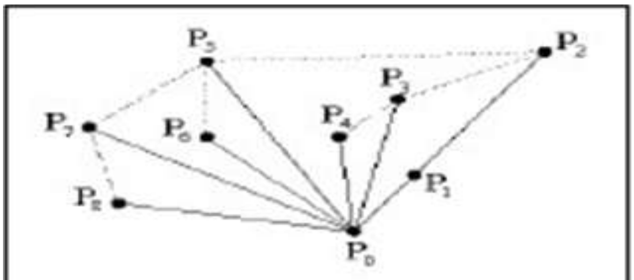


figure (3) label of sorting points

Step7: Remove right turn point If the inner angle formed by Pa, Pb,Pc is greater than or equal to 180° then we eliminate the point labeled with Pb. Setpoint Pb to be point Pa. Set point Pa to the previous point in the sequence (in this case P5) [13]. As showing in figure(4).

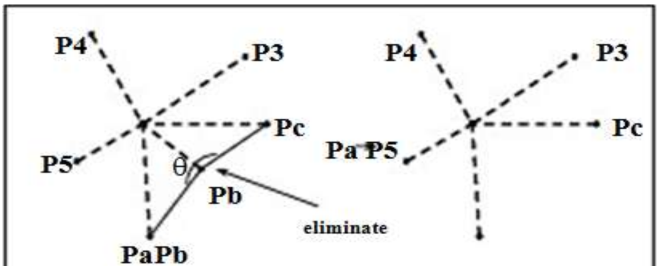


figure (4) right turn point

Step8: Add to CH the left turn points. If the inner angle formed by Pa, Pb, Pcfrom before is less than180° then No points are discarding. Each of Pa,Pb and Pc are advances forward one point [14] as shown in figure (5).

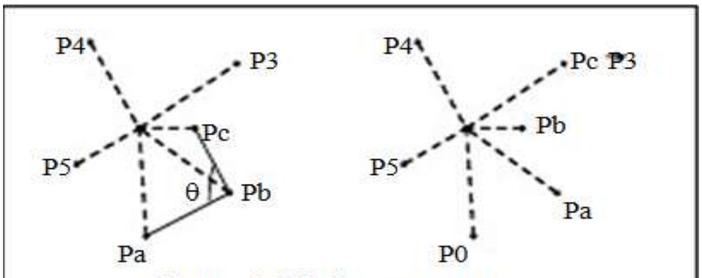


figure (5) left turn point

Step9: The Algorithm continues by repeating step 7 until Pb=P0. At this case,the algorithm stops and only the points of the convex hull CH remain as output [15].

Finish

3-2 Mergsort algorithm

Merge sort depend on divide and conquer strategy to sort the points of list or matrix this means Mergesort recursively divides the list of points into smaller sub lists andsort them individually and then merge these sub lists to form a complete sorted list[16].Conceptually, mergesort works as follows:

Mergesort algorithm

Input: Set of angles (list) between P0 and others points P.

Output: The Set of angles sorted in counter clockwise order.

Start

Step1: Finding the median (mid) of list .

Mid = (first + list) div 2 ------------------- (3)

Where:- mid : index of median point. first , last: index of first and lastpoint respectively.

Step2: Divide the unsorted list into two sub lists of about half the size.

Mergesort (list, first, Mid)----------------- (4)

Mergesort (list, Mid+1,last)

Step3: Divide each of the two sub lists recursively until we have list sizes of length one point.

Step4: Merge the two sorted sub lists back into one sorted list.

Finish

4. Diameter-finding and width-finding: Rotating calipers

Before explain the Diameter-finding algorithm it is beneficial to describewhat is the Diameter and width of a set of two dimension points, and current someterminology [6]. We call any two edges of the convex polygon CH of a set of two dimension pointsS an antipodal vertex-vertex (V-V) pair if parallel lines of convex polygonof S contain these vertices. Similarly, we can define vertex-edge (V-E) pairs.Note that these are the only probable combinations to find width or diameter of convex hullpolygon, since a line of support of S must cross CH either at onlyvertex, or along one of CH edges [3], as in figure(6).

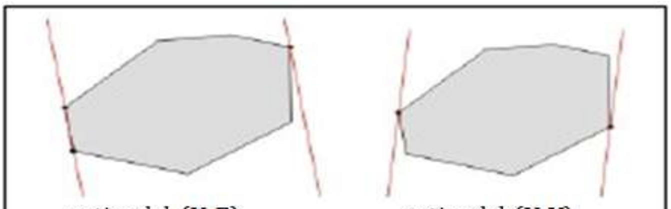


figure (6) antipodal pairs

4.1 Diameter-finding algorithm

The diameter of a convex polygon defined as the maximum distancebetween parallel lines of the polygon. Note that the pair of points determiningthe diameter of a convex polygon CH does not belong to the interior of CH, [2].The search should focus on the external. In fact, only antipodal pairs of pointsshould checked, for the diameter is the highest distance between two parallel linesof polygon [4].

Diameter-finding algorithm

Input: Set points CH= (S0 , S1 , S2 ,…) of convex polygon vertices.

Output: Antipodal pair has maximum distance.

Start

Step1: Compute the polygon's finishing points in the y direction. Callthem ymin and ymax.

Step2: Construct two horizontal lines of support through ymin and ymax.Since this is already an antipodal pair, we compute the area of triangle ofymin,ymax and S vertices ,then keepas maximum [17].

Given the coordinates of the three vertices of a triangle ABC, the area is given by

$$area = \left| \frac{Ax(By - Cy) + Bx(Cy - Ay) + Cx(Ay - By)}{2} \right|$$

where A_x and A_y are the x and y coordinates of the point A etc. ----------------- (5)

Step3: Use Rotating calipers to generate antipodal pairs with an edge of the convexPolygon.

Step4: Compute the new area and compare to old maximum, then update ifnecessary.

Step5: Repeat steps 3 and 4 until the antipodal pair reachedis (ymin,ymax) again.

Step6: Report or output the antipodal pair with maximum distance as the diameter pair.

Finish

4.2 Width-finding algorithm

The width of a convex hull polygon distinct as the smallest distance betweentwo parallel lines of polygon [1]. Note not all directions of polygon need to emphasis.Presume a polygon vertices were obtain, alongside with two parallel lines of convex polygon.If neither of the parallel lines coincides with an edge, then it is continuously possible to rotatethem to minimize the distance between them. Therefore, two parallel lines ofCH do not measure the width except one of them concurs with an edge[3].

Width-finding algorithm

Input: Set points CH= (S0 , S1 , S2 ,…) of convex polygon vertices.

Output: Antipodal pair has minimum distance.

Start

Step1:compute the finishing point of polygon in the y orientation,denoted as ymin and ymax.

Step2: Construct two parallel lines of CH through ymin and ymax. Ifone (or both) of the Lines tangent with an edge, then a antipodalvertex-edge or edge-edge pair is detected, compute the area oftriangle(ymin,ymax ,S) vertices, and save as the minimum distance.

Step3: Rotate the lines until one is attached with an edge of the polygon. A newantipodal pair is detected. Compute the new distance between thelines, compare to old minimum, and update the distance if it necessary.

Step4: Repeat steps 3 until the lines of calipers have reached their originalhorizontal position.

Step5: Report or output the antipodal pair with minimum distance as the width ofpolygon.

Finish

5. implementation of programming system and results

In this paper, the rotating calipers algorithm to find Diameter and Widthof convex polygon starts with an unordered set of points defined by Cartesiancoordinates. Each point has a position on the X-axis and Y-axis.To illustrate thealgorithm we'll start with example of generate 500 points.

5-1 Implementation of programming system

First: Generate the two dimensional points with maximum number not exceeding 10000 points and draw them using point drawing instruction according to programming language. Then the algorithm compute or take starting point to be anchor point which represent the maximum Y- Coordinate, after that compute theangles between the anchor point to all other points from the horizontal line using atan2 function in the equation(1) as shown in figure (7).

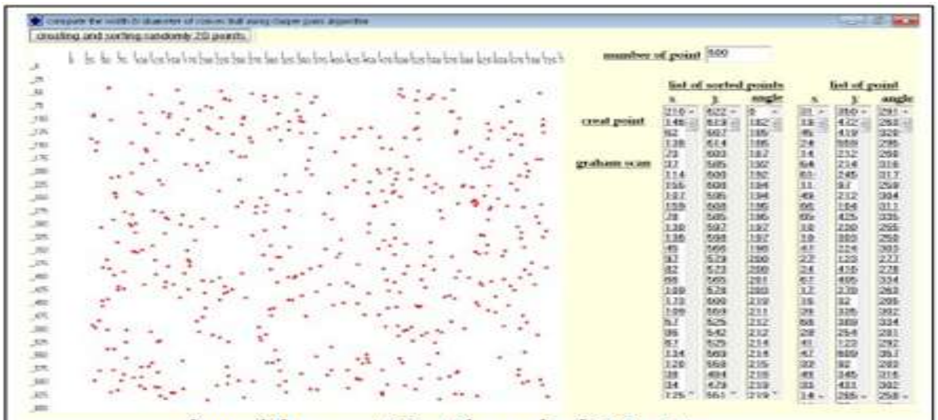


figure (7) generate & sort by angle of 500 point

Second: Sort the points by angles in counter clockwise order using equation number(3,4) represents the mergsortalgorithmsteps , in the same time the two dimension points are also rearrange according to value of angle belongs to eachpoint. Finally we draw line between anchor point and all othersorted points to show thearrangement of points as shown in figure (8)

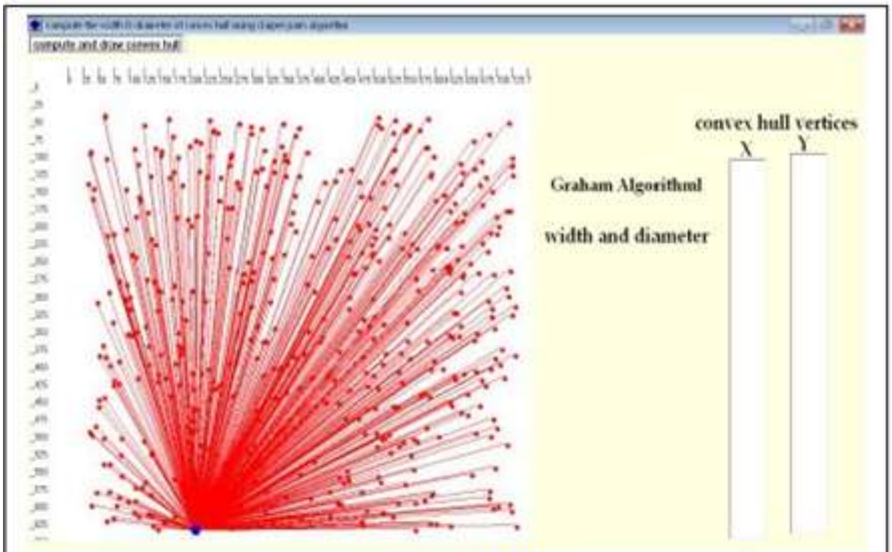


Figure (8) sort the point

Third: Build the convex hull using Graham scan algorithm as in figure (9).

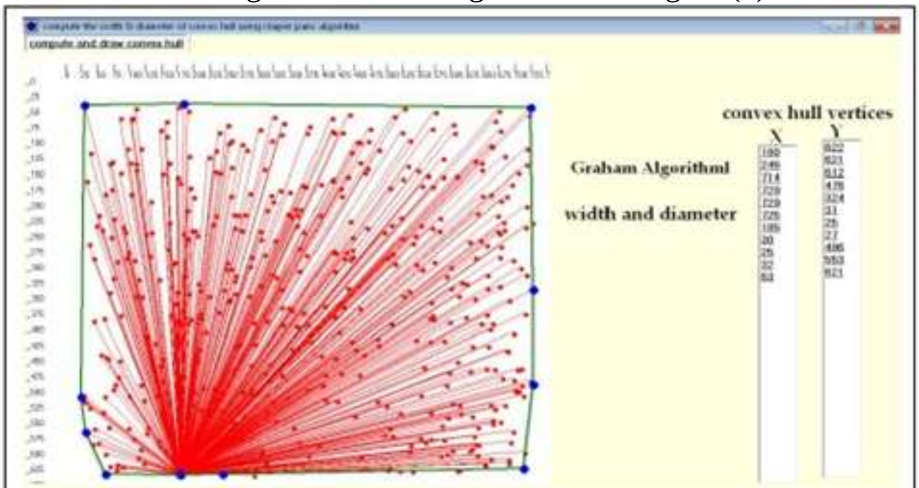


Figure (9) Graham convex polygon

Forth: Compute the Width & Diameter of convex polygon using rotatingcalipers algorithm as in figure (10). Diameter in red and width in blue color.

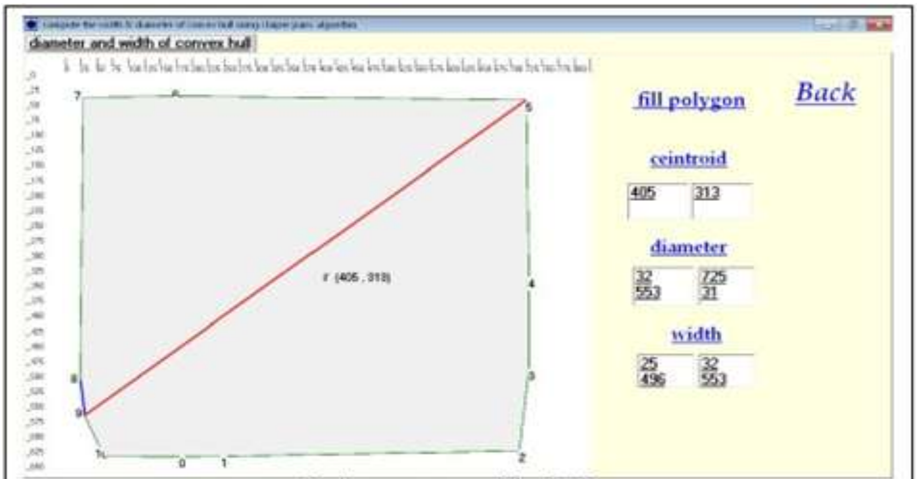


Figure (10) Diameter & Width of polygon

5-2 Results

From the practical implementation of proposed software system, we findthe number of convex hull polygon vertices for variety of two dimension data sizeand puts the resulting in one table as bellow

#OF POINTS	#HULL POINTS
5	4
10	7
50	10
100	11
300	14
500	15
600	15
800	16
1000	15
1500	17
10000	20

6. Conclusion

1. The experimental results reveal that convex hull points are only a small subset of all the randomly two dimension points, so we recommends computing the hull and then doing all antipodal pairs distances computation.
2. The Random function use in generating two dimension points is a speedy function in having randomtwo dimension points.
3. The rotating calipers algorithm is fast, easy and common tool for solving computational geometric problems.
4. Mergesort algorithm used to sort floating points represent angles of points is fastalgorithm and influences the runtime of Graham scanning procedure.
5. Rotating Calipers a highly useful and adaptable tool that deserves further research.
6. The Graham Scan algorithm has the optimal worst case complexity whennot taken account output sensitivity. The basic concept is that we take aninfinity point, sort all the other points angularly in O(n log n), with a mass in linear time to compute the convex hull polygon. Since wesort in this algorithm, we are bounding by the output insensitive complexity.
7. The convex hull is one of the most basic constructions of computationalgeometry, used as a stepping stone to solving much firmer and morecomplex problems.
8. There is a fact that optimal algorithms for building convex hull polygon already existfor both n-space with 2D or 3D point sets.

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