Compactly ω-closed set and compactly ω-k-closed set

المجموعة المغلقة ω رصاً والمجموعة المغلقة ω -K رصاً

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Abstract :

In this paper ,we introduce definition of compactly ω-closed set and obtain some fundamental properties for this concept . Moreover, we also study the definition of compactly ω-k-closed set and prove some results which are relate to this subject . Also ,we give the relation between these concepts .

Key Words: ω-open set ,ω-closed set ,Compactly ω-closed ,Compactly ω-k-closed set , St-ω-compact function ,ω-compact function .

الخلاصة:

في هذا البحث , نُقدّمُ تعريفَ المجموعة المغلقة-ω رصاً ونحصل على بعض الخواص الأساسية لهذا المفهوم . علاوة على ذلك، نَدْرسُ تعريفَ المجموعة المغلقة k-ω رصاً أيضاً ونُثبتُ بَعْض النَتائِجِ التي تَتعلّقُ بهذا الموضوعِ. أيضاً، نَعطي العلاقةَ بين هذه المفاهيمِ.

0.Introduction

In 1982 , Hdeib defined [1]ω-closed and ω-open sets as follows :A sub set of a space is called ω-closed if it contains all its condensation point :A point $x \in X$ is called a condensation point of A if for each $U \in T$,the set $U \cap A$ is un countable. The complement of an ω-closed set is said to be ω-open .Equivalently a sub set W of a space (X,T) is ω-open if and only if for each $x \in W$,there exists $U \in T$ such that $x \in U$ and U-W is countable . In 1989 , Hdeib [2] introduced ω-continuity as a generalization of continuity as follows :A function $f:(X,T) \to (Y,\sigma)$ is called ω-continuous if the inverse image of each open set is ω-open set .In the year 2003, the authors [3] proved that the family of all ω-open sets in a space (X,T) forms a topology X finer than that T.

This paper contains three sections :In section one of this paper we review the basic concepts that we needed . In section two ,we introduce the compactly ω-closed set and basic properties of this set . In section three ,we study the relation between compactly ω-closed set and compactly ω-k-closed set and review some properties of the last notion .

Throughout this paper by a space we mean a topological space .The closure of A and the interior of A will be denoted by CL(A) and INT(A) ,respectively.

1.Preliminaries

The section includes some basic definitions ,theorems ,propositions and remarks that we need for our main results in this paper .

Definition(1-1):[1] Let (X,T) be a space ,$W \subset X$ .A set W is said to be ω-open if for each $x \in W$,there exists $U \in$ T with $x \in U$ and U-W is countable .The complement of ω-open set is called ω-closed set .The ω- closure and ω-interior of a set W ,will be denoted by $CL_{\omega}(W)$ and $INT_{\omega}(W)$ resp.,are defined by

$$CL_{\omega}(W) = \cap \{F \subset X/F \; is \; \omega - closed \; and \; W \subset F\}$$
$$INT_{\omega}(W) = \cup \{G \subset X/G \; is \; \omega - open \; and \; G \subset W\}$$

Remark (1-2):[3] In a space (X,T) :

i. Every open set is an ω-open set .
ii. Every closed set is an ω-closed set.
iii. A sub set U of (X,T) is ω-open (ω-closed) set if and only if its open (closed)set in (X, T_{ω}) .

Definition (1-3):[4]Let X be a space .We say that a subset A of X is ω-compact if for each cover of ω-open sets from X contains a finite sub cover for A .If A=X we say that X is ω-compact space .

Example (1-4): Every finite subset of any space is an ω-compact.

Remark (1-5):

i. The space (X,T) is ω-compact if and only ,if the space (X, T_{ω}) is compact , [4].
ii. Every ω-compact space is compact ,[5].

Lemma (1-6):[6] In any space the intersection of a compact sub set with closed subset is compact .

Proposition (1-7): In any space X ,the intersection of any ω-closed with any ω-compact set is ω-compact .

Proof :

Let A,B be an ω-closed , ω-compact set ,respectively of a space (X,T).Thus A,B be closed , compact set ,respectively of a space (X, T_{ω}) , then by Lemma (1-6) , $A \cap B$ is a compact set in (X, T_{ω}) ,then by Remark (1-5,i) $A \cap B$ is an ω-compact in (X,T).

Theorem (1-8):[6]

i. Every compact subset of a T_2 space is closed .
ii. Closed sub sets of compact sets are compact .

Proposition (1-9):

i. Every ω-closed subsets of an ω-compact space set is ω-compact .
ii. Every ω-compact sub set of an ω-T_2 space is ω-closed .

Proof:

i.Let A be ω-closed sub set of ω-compact space (X,T) ,then by Remark (1-5,i) A is closed subset of compact (X,T_{ω}) ,then by Theorem (1-8,ii) , A is a compact subset of (X,T_{ω}) ,therefore by Remark (1-5,i) A is an ω-compact subset of (X,T).

ii.Let A be ω- compact subset of an ω-T_2 space ,then A be compact subset of an T_2 space by using(1-9,i) ,A is closed subset ,then A is ω-closed subset .

Definition (1-10): Let X and Y be a spaces ,the function $f: X \to Y$ is said to be :

i. ω-continuous if for every open sub set A of Y ,$f^{-1}(A)$ is ω-open in X ,[5].
ii. ω-irresolute if for every ω-open sub set B of Y ,$f^{-1}(B)$ is ω-open in X ,[3].
iii. ω-compact if for every ω-compact sub set K of Y,$f^{-1}(K)$ is compact in X.

Definition (1-11):[3]A space X is said to be ω-T_2 if for $x, y \in X$ such that $x \neq y$,there exists disjoint ω-open sets U and V such that $x \in U \; and \; y \in V$.

Proposition (1-12):[7]A space (X,T) is ω-T_2 if and only ,if (X, T_ω) is T_2 .

Remark(1-13): Let X be a space and Y be a sub space of X such that $A \subseteq Y$,if A is an ω-open (ω-closed) sub set in X then A is an ω-open (ω-closed) in Y .

Proposition (1-14): Let X be a space and Y be an ω-open set of X ,if A an ω-open set in Y then A is an ω-open set in X .
Proof:

Let $x \in A$,since A is an ω-open in Y then there exists an open set W in Y contain x such that W-A is countable set .Since W is an open set in Y ,then $W = U \cap Y$ for some open set U in X , $U \cap Y$ is an ω-open set in X ,since $x \in U \cap Y$ and W is ω-open set in X contain x ,then there exists an open set V in X contain x such that V-W is countable set .
Since V-A⊆ (V-W)⋃(W-A) and (V-W)⋃(W-A) is countable set ,then V-A is countable set .So A is an ω-open set in X .

Proposition (1-15): Let X and Y be spaces and $f: X \to Y$ be function then if f is ω-irresolute function ,then an image $f(A)$ of any ω-compact A in X is an ω-compact set of Y .

Proof :

Let $\{V_\lambda \backslash \lambda \in \Delta\}$ be an ω-open cover of $f(A)$,then $\{f^{-1}(V_\lambda) \backslash \lambda \in \Delta\}$ is an ω-open cover of A. Since A is ω-compact in X ,then A has a finite sub cover say $\{ f^{-1}(V_{\lambda_i})\}_{i=1}^{n}$,i.e. $A \subseteq \bigcup_{i=1}^{n} f^{-1}(V_{\lambda_i})$,hence $f(A) \subseteq \bigcup_{i=1}^{n} V_{\lambda_i}$ .Therefore $f(A)$ is ω-compact of Y.

Proposition(1-16): Let X be a space and $A \subseteq X$,$x \in X$ then $x \in CL_\omega(A)$ if and only ,if there exists a net $(\chi_d)_{d \in D}$ in A and $\chi_d \overset{\omega}{\to} x$.

Proof :

Let $x \in CL_\omega(A)$,then for all $U_x \in T_\omega$, $x \in U_x$,$U_x \cap A \neq \phi$.Notic that $(N_\omega(x) , \geq)$ is a directed set ,such that for all $U_1, U_2 \in N_\omega(x)$, $U_1 \geq, U_2$ if and only if $U_1 \subseteq U_2$,now since $U_x \cap A \neq \phi$ for all $U_x \in N_\omega(x)$,thus there exists $\chi_{U_x} \in U_x \cap A$ for all $U_x \in N_\omega(x)$ .Hence $(\chi_{U_x})_{U_x \in N_\omega(x)}$ is a net in A .To prove that $\chi_{U_x} \overset{\omega}{\to} x$ .Let $U_0 \in N_\omega(x)$ and let $U \geq U_0$,thus $\chi_U \in U \subseteq U_0$,then $\chi_U \in U_0$ .Hence $\chi_{U_x} \overset{\omega}{\to} x$.

Conversely ,suppose that there exists a net $(\chi_d)_{d \in D}$ in A such that $\chi_d \overset{\omega}{\to} x$ (To prove that $x \in CL_\omega(A)$) ,let $U \in N_\omega(x)$ .Since $(\chi_d)_{d \in D}$ is eventually in every ω-neighborhood of x ,then there exists $d_0 \in$ D such that $\chi_d \in U$ for all $d \geq d_0$ but $\chi_d \in A$ for all $d \in$D ,thus $U \cap A \neq \phi,$ for all $U \in N_\omega(x)$ .Hence $x \in CL_\omega(A)$.

Proposition(1-17): Let Y be an ω-open sub space and K⊆ Y ,then K is an ω-compact set in Y if and only ,if K is an ω-compact set in X.

Proof :

Let $\{U_\lambda\}_{\lambda \in \Delta}$ be an ω-open cover in X of K and $V_\lambda = U_\lambda \cap Y$ for all $\lambda \in \Delta$ .Then V_λ is an ω-open in X for all $\lambda \in \Delta$,but $V_\lambda \subseteq Y$,thus by Remark (1-13) , V_λ is an ω-open cover in Y for all $\lambda \in \Delta$ .Since $K \subseteq \bigcup_{\lambda \in \Delta} V_\lambda$,then $\{V_\lambda\}_{\lambda \in \Delta}$ is ω-open cover in Y of K and by hypothesis this cover has finite sub cover of K. Hence the cover $\{U_\lambda\}_{\lambda \in \Delta}$ has finite sub cover of K .Thus K is an ω-compact set in X

Conversely ,let K is an ω-compact set in X (To prove that K is an ω-compact set in Y).Let $\{H_\alpha \backslash \alpha \in \Delta \}$ be ω-open cover in Y of K .Since Y be an ω-open sub space of X ,then by Proposition (1-14) , $\{H_\alpha \backslash \alpha \in \Delta \}$ is ω-open cover in X of K .Then by hypothesis there exists finite sub cover such that $K \subseteq \bigcup_{i=1}^{n} H_{\alpha_i}$.Hence K is an ω-compact set in Y .

2.Compactly ω-closed set

In this section ,we introduce a new notion of ω-closed set which is compactly ω-closed set .Also ,we give some proposition ,examples and theorems about this subject .

Definition (2-1): Let X be a space .A sub set W of X is called a compactly ω-closed if for every ω-compact set K in X ,W $\cap K$ is ω-compact .

Example (2-2):

i. Every finite sub set of a space X is compactly ω-closed set.
ii. Every sub set of a discrete space is compactly ω- closed set .

Proposition (2-3): Every ω-closed sub set of a space X is compactly ω-closed set.
Proof:
Let A be an ω-closed sub set of a space X and K be an ω-compact set in X .Then by Proposition (1-7)A ∩K is a ω-compact set .Thus A is a compactly ω-closed set.

The converse of Proposition (2-3) is not true in general as the following example shows :

Example(2-4): Let X=$\mathbb{Q}^c$,A=$\mathbb{Q}^c - \{\sqrt{2}\}$ be a sets and let (X,T) is indiscrete space where $\mathbb{Q}^c$ is the set of irrational numbers .Then A is compactly ω-closed set ,but A is not ω-closed set ,since $A^c \notin$ T_ω .

Theorem(2-5): Let X be a T_2 space .A sub set A of X is compactly ω-closed if and only ,if ω-closed set .
Proof :
Let A be a compactly ω-closed set in a space X and let $x \in CL_\omega(A)$,then by Proposition (1-16) there exists a net $(\chi_d)_{d\in D}$ in A such that $\chi_d \overset{\omega}{\rightarrow} x$,then F=$\{\chi_d, x\}$ is an ω-compact set .Since A is an compactly ω-closed , A ∩F is an ω-compact .So, by Proposition (1-9,ii) A ∩F is an ω-closed set . Since $\chi_d \overset{\omega}{\rightarrow} x$ and $\chi_d \in$ A ∩F ,then by proposition (1-16) $x \in$ A ∩F $\rightarrow$ $x \in A$ .Hence $CL_\omega(A) \subseteq A$, therefore A is ω-closed set .

Conversely ,by Proposition (2-3) .

Before we prove the next propositions ,we state the following notion of a strong ω-compact function.

Definition (2-6): Let X and Y be spaces ,the function $f: X \rightarrow Y$ is called a strong ω-compact function (st-ω-compact function) if the inverse image of each ω-compact set in Y is ω-compact set in X .

Example (2-7): Every constant function X into Y is st-ω-compact function .

Proposition (2-8): Let $f: X \rightarrow Y$ be an ω-irresolute ,st-ω-compact and one to one ,function .Then A is compactly ω-closed set in X if and only ,if $f(A)$ is compactly ω-closed set in Y .
Proof :
Let A be compactly ω-closed set in X and let K be an ω-compact set in Y .Since f is st-ω-compact function then $f^{-1}(K)$ is ω-compact set in X .So, A $\cap f^{-1}(K)$ is an ω-compact set. Then by proposition (1-15) $f($A $\cap f^{-1}(K))$ is ω-compact set in Y but $f($A $\cap f^{-1}(K)) = f(A)$ ∩K ,then $f(A)$ ∩K is ω-compact set .Hence $f(A)$ is compactly ω-closed set .

Conversely, let $f(A)$ be a compactly ω-closed set in Y and let K be ω-compact set in X .Since f is ω-irresolute function ,then by proposition (1-15) $f(K)$ is ω-compact set in Y ,so$f(A) \cap f(K)$ is ω-compact set in Y .
Since f is st-ω-compact function, then$f^{-1}(f(A) \cap f(K)) = f^{-1}(f(A)) \cap f^{-1}(f(K))$ is ω-compact set in X .Since f is one to one to one function then A= $f^{-1}(f(A))$ and K=$f^{-1}(f(K))$.
Thus A ∩K=$f^{-1}(f(A)) \cap f^{-1}(f(K))$ .Hence A ∩K is ω-compact set in X .

Therefore A is compactly ω-closed set in X .

Proposition (2-9): Let B be an ω-open sub space of space X .Then B is compactly ω-closed set if and only ,if the inclusion function $i_B: B \to X$ is st-ω-compact .

Proof :

Let K be an ω-compact in X ,then B$\cap K$ is an ω-compact in X thus by Proposition(1-17) , B$\cap K$ is an ω-compact in B but ${i_B}^{-1}(K)$= B$\cap K$,then ${i_B}^{-1}(B)$ is ω-compact in B .Hence $i_B: B \to X$ is st-ω-compact function .

Conversely ,let K be ω-compact in X . Since i_B is a st-ω-compact ,then ${i_B}^{-1}(B)$ is an ω-compact in B .Thus by Proposition(1-17) ${i_B}^{-1}(K)$ is an ω-compact set in X but ${i_B}^{-1}(K)$= B$\cap K$,then B$\cap K$ is an ω-compact set in X ,for every ω-compact K in X .Therefore B is a compactly ω-closed set .

3.The relationship between compactly ω-K-closed set and compactly ω-closed set.

In this section of the paper we introduce definitions of compactly ω-K-closed set and its relation with compactly ω-closed .

Definition (3-1): Let X be a space .Then a sub set A of X is called compactly ω-K-closed if for every ω-compact set K in X , A ∩K is ω-closed set .

Example (3-2): Let X=$\mathcal{R}$ and (X,T) is co finite space where $\mathcal{R}$ is the set of real numbers .Then every finite set of this space is compactly ω-K-closed .

Proposition (3-3): Every compactly ω-K-closed sub set of a space X is ω-closed set .

Proof:

Let A compactly ω-K-closed sub set of a space X and let $x \in CL_\omega(A)$,then by Proposition(1-16) there exists a net $(\chi_d)_{d\in D}$ in A such that $\chi_d \overset{\omega}{\to} x$ .Then F=$\{\chi_d, x\}$ is an ω-compact set .Since A an compactly ω-K-closed set then A ∩F is ω-closed set .Since $\chi_d \overset{\omega}{\to} x$ and $\chi_d \in$ A ∩F ,then by Proposition (1-16) $x \in$ A ∩F $\to$ $x \in$ A .Hence $CL_\omega(A) \subseteq A$,therefore A is ω-closed set .

The converse of proposition (3-3) is not true in general as the following example shows :

Example (3-4) : X=$\mathbb{Q}^c$,A=$\mathbb{Q}^c - \{\sqrt{2}\}$ be a sets and let T={ ϕ , $\mathbb{Q}^c$, $\{\sqrt{2}\}$ } be a topology on X . Notice that A is ω-closed set ,but A is not compactly ω-K-closed set ,since A ∩B is not ω-compact set for any ω-compact set B in X .

Proposition (3-5): Every compactly ω-K-closed sub set of a space X is compactly ω-closed set .

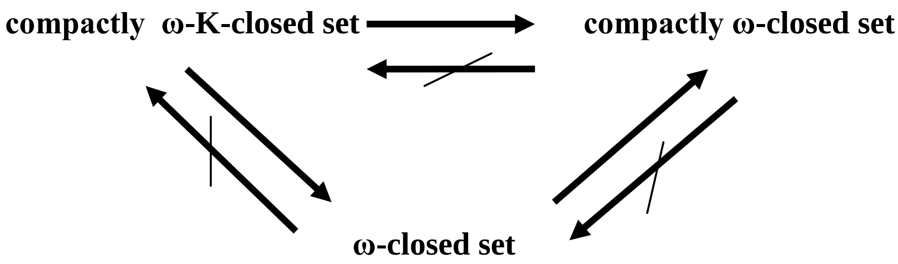
Proof:

Let A compactly ω-K-closed sub set of a space X and let K be an ω-compact set in X ,then A ∩K is ω-closed set .Since A ∩K $\subseteq K$ and K is ω-compact set in X . Then by Proposition (1-9,i) A ∩K is ω-compact set .Therefore A is compactly ω-closed set .

Note that the opposite direction of proposition (3-5) is not true .The following example shows that

Example (3-6) : X=$\mathbb{Q}^c$,A=$\mathbb{Q}^c - \{\sqrt{2}\}$ be a sets and let (X,T) is indiscrete space where $\mathbb{Q}^c$ is the set of irrational numbers .Then A is compactly ω-closed set ,but A is not compactly ω-K-closed set ,since A ∩B is not ω-compact set for any ω-compact set B in X .

Remark (3-7): From the propositions(2-3,3-3,3-5), we get the following diagram ,and none of its implications is reversible and the Examples (2-4,3-4,3-6) shows that ,respectively .



The following theorem shows that under certain condition , ω-closed , compactly ω-closed and compactly ω-K-closed sets are equivalent .

Theorem (3-8): Let X be T_2 -space and A is a sub set of X .Then the following statement are equivalent :

i. A is compactly ω-K-closed set .
ii. A is compactly ω-closed set.
iii. A is ω-closed set.

Proof :

(i→ii) By proposition (3-5) .

(ii →i) Let A be a compactly ω-closed subset of X and K be an ω-compact set in X. Then A ∩K is ω-compact set .Since X is T_2 -space , Then by proposition (1-9,ii) , A ∩K is ω-closed set .Hence A is a compactly ω-K-closed set .

(iii→i) Let A be an ω-closed sub set of a space X and K be an ω-compact set in X .Then by Proposition (1-7)A ∩K is a ω-compact set .Then by Proposition (1-9,ii) A ∩K is a ω- closed set. Hence A is a compactly ω-K-closed set.

(i→ iii) By Proposition (3-3) .

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