

On Regular Generalized α - Quotient Mappings in Topological Spaces

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حول التطبيقات ألقسمه المنتظمة المعممة من النمط - α في الفضاءات التبولوجية

مقدم من قبل

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Abstract :

In this work , we introduce a new type of quotient mappings namely (regular generalized α - quotient maps ,strongly regular generalized α -quotient maps and regular generalized α^* - quotient maps) in topological spaces, and we study some of their properties and relations among them.

Key word :

α -quotient map , generalized α - quotient map ,strongly regular generalized α -quotient maps .

المستخلص :

في هذا العمل، قدمنا نوع جديد من التطبيقات القسمة تدعى (تطبيقات ألقسمه المنتظمة المعممة من النمط - α ، تطبيقات ألقسمه القوية المنتظمة المعممة من النمط- α و تطبيقات القسمة المنتظمة المعممة من النمط - α^*) في الفضاءات التبولوجية ، و درسنا بعض من خواصها والعلاقات بينها.

1-Introduction :

In 1996,Thivagar [7] introduced the notion of quotient mappings and α - quotient mappings .Recently, Ravi et al [4],[5] they introduced and discussed the concepts α gs- quotient maps and \hat{g} - quotient maps in topological spaces.

vadivel and vairamanickam [8] introduce the concept $rg\alpha$ -closed and $rg\alpha$ -open sets in topological spaces . They used these sets to introduce $rg\alpha$ -continuous functions types and investigated some of their basic properties [9].

This paper to introduce and investigate a new class of mappings namely ($rg\alpha$ - quotient maps ,strongly $rg\alpha$ -quotient maps, $rg\alpha^*$ - quotient maps) . Also, the relationship between weak and strong form of $rg\alpha$ - quotient maps have been established . Moreover, we study and proved some of their properties .

Throughout this paper (X, τ) and (Y, σ) (or simply X and Y) represent non-empty topological spaces. For a subset A of a space X , $cl(A)$, $int(A)$ and A^c denoted the closure of A , the interior of A and the complement of A in X respectively.

2-Preliminaries:

Some definition and basic concepts have been given in this section .

Definition (2-1):

A subset A of a space (X, τ) is said to be a :

- 1- **semi open set** [1] if $A \subseteq cl(int(A))$ and **semi closed set** if $int(cl(A)) \subseteq A$.
- 2- **α -open set** [2] if $A \subseteq int(cl(int(A)))$ and **α -closed set** if $cl(int(cl(A))) \subseteq A$.
- 3- **regular open set** [6] if $A = int(cl(A))$ and **regular closed set** if $A = cl(int(A))$.

Definition(2-2):

The intersection of all α -closed subsets of (X, τ) containing A is called the **α - closure** [2] of A and is denoted by $acl(A)$.

Definition(2-3): A subset A of a space (X, τ) is said to be a :

- 1- **Regular generalized closed set** (briefly, rg-closed) [3] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is regular open set in X .
- 2- **regular α -open set** (briefly $r\alpha$ -open) [8] if there is a regular open set U such that $U \subset A \subset acL(U)$. The family of all regular α -open sets of (X, τ) is denoted by $R_{\alpha}O(X)$.
- 3- **regular generalized α -closed set** (briefly, $rg\alpha$ -closed) [8] If $acL(A) \subset U$ whenever $A \subset U$ and U is regular α - open in (X, τ) . The set of all $rg\alpha$ -closed set in (X, τ) denoted by $RG \alpha C(X)$.
- 4- **regular generalized α -open** (briefly, $rg\alpha$ -open) [8] in (X, τ) . If A^c is $rg\alpha$ -closed set in (X, τ) , The family of all $rg\alpha$ -open sets in (X, τ) denoted by $RG \alpha O(X)$.
- 5- **\hat{g} -closed set (= w-closed)** [11] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an semi-open set in (X, τ) . The complement of \hat{g} -closed set is called **\hat{g} - open set**.

Remark(2-4) :

In [2],[11] and[8] , proved that:

- 1- Every open set in a space (X, τ) is α -open set and \hat{g} -open set in (X, τ) .
- 2- Every \hat{g} -open (=w-open) set in a space (X, τ) is $rg\alpha$ -open set in (X, τ) .
- 3- Every open(resp , α -open) set in a space (X, τ) is $rg\alpha$ -open set in (X, τ) .
- 4- Every $rg\alpha$ -open set in a space (X, τ) is rg-open set in (X, τ)

The converse of the above Remark need not be true ,as seen from the following example:

Example(2-1):

(i) Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{a, c\}\}$. Then the set $A = \{a, b\}$ is α -open set in X ,but A is not open set in (X, τ) .

(ii) Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$. Then the set

$A = \{d\}$ is a $rg\alpha$ -open set in (X, τ) , but A is not open (α -open set and rg -open set) in X .

(iii) Let $X = \{a, b, c, d, e\}$, $\tau = \{X, \Phi, \{a\}, \{d\}, \{e\}, \{a, d\}, \{a, e\}, \{d, e\}, \{a, d, e\}\}$. Then $A = \{a, c, d, e\}$ is a $rg\alpha$ -open set in (X, τ) , but A is not \hat{g} -open set in (X, τ) .

Definition(2-5):

A map $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

1- **α -continuous** [2] if the inverse image of every open (closed) set in (Y, σ) is an α -open (α -closed) set in (X, τ) .

2- **$rg\alpha$ -continuous** [9] if the inverse image of every open (closed) set in (Y, σ) is an $rg\alpha$ -open ($rg\alpha$ -closed) set in (X, τ) .

3- **$rg\alpha$ -irresolute** [9] if the inverse image of every $rg\alpha$ -open ($rg\alpha$ -closed) set in (Y, σ) is an $rg\alpha$ -open ($rg\alpha$ -closed) set in (X, τ) .

4- **Strongly $rg\alpha$ -continuous** [9] if the inverse image of every $rg\alpha$ -open ($rg\alpha$ -closed) set in (Y, σ) is an open (closed) set in (X, τ) .

5- **\hat{g} -continuous** (=w-continuous) [8] if the inverse image of every open (closed) set in (Y, σ) is \hat{g} -open (\hat{g} -closed) set in (X, τ) .

Definition(2-6): [10]

A map $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

1- $rg\alpha$ -open if $f(A)$ is $rg\alpha$ -open set in (Y, σ) for every open set A in (X, τ) .

2- $rg\alpha^*$ -open if $f(A)$ is $rg\alpha$ -open set in (Y, σ) for every $rg\alpha$ -open set A in (X, τ) .

Definition(2-7):

A space (X, τ) is said to be a **$T^{*1/2}$ -space** [3] if every rg -closed set in X is closed.

Definition(2-8):

A surjection map $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be:

1-**quotient map** [11], provided a subset A of (Y, σ) is open in (Y, σ) if and only if $f^{-1}(A)$ is an open in (X, τ) .

2- **α - quotient map** [11] if f is α -continuous and $f^{-1}(A)$ is an open in (X, τ) implies A is α -open set in (Y, σ) .

3- **\hat{g} - quotient map** [6] if f is \hat{g} -continuous and $f^{-1}(A)$ is an open in (X, τ) implies A is \hat{g} -open set in (Y, σ) .

Remark(2-9):

In [9], proved that:

1-Every continuous (α -continuous) map is $rg\alpha$ -continuous.

2-Every $rg\alpha$ -irresolute map is $rg\alpha$ -continuous.

3-Every \hat{g} -continuous map is $rg\alpha$ -continuous.

The converse of Remark(2-9) need not be true , as seen from the following Examples:

Example(2-2):

(i) Let $X=Y=\{a, b, c\}$ with topology $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \emptyset, \{a\}\}$. Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a map defined by $f(a)=c, f(b)=b$ and $f(c)=a$. Then f is $rg\alpha$ -continuous map , but f is not continuous (α -continuous) map . Since for open set $\{a\}$ in (Y, σ) . $f^{-1}(\{a\}) = \{c\}$ is not open and is not α -open set in (X, τ) .

(ii) Let $X=Y=\{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{a, c\}\}$.

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a map defined by $f(a)=a, f(b)=c$ and $f(c)=b$. Then f is $rg\alpha$ -continuous map , but f is not $rg\alpha$ -irresolute . Since for $rg\alpha$ -open set $\{a, b\}$ in (Y, σ) $f^{-1}(\{a, b\}) = \{a, c\}$ is not $rg\alpha$ -open set in (X, τ) .

(iii) Let $X=Y=\{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$ and $\sigma = \{Y, \emptyset, \{a\}\}$.

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a map defined by $f(a)=c, f(b)=f(c)=a$. Then f is $rg\alpha$ -continuous map , but f is not \hat{g} - continuous map . Since for open set $\{a\}$ in (Y, σ) . Then $f^{-1}(\{a\}) = \{c\}$ is not \hat{g} - open set in (X, τ) .

Remark(2-10):

In [7]. Every quotient map is α - quotient map , but is not conversely.

Example(2-3):

Let $X=\{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{b, c\}\}, Y=\{1, 2\}$ and $\sigma = \{Y, \emptyset, \{a\}\}$.

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a map defined by $f(a)=2, f(b)=f(c)=1$. Then f is α -quotient map , but f is not quotient map . Since $f^{-1}(\{2\}) = \{a\}$ is an open set in (X, τ) . But $\{2\}$ is not open set in (Y, σ) .

3- On Some Regular Generalized α - Quotient Mappings Types :

In this section we introduce a new types of quotient maps which are[regular generalized α -quotient maps , strongly regular generalized α -quotient maps and regular generalized α^* -quotient maps], and studying the relations between them . Also, we give propositions about the compositions of these maps .

Definition(3-1) :

A surjection map $f: (X, \tau) \rightarrow (Y, \sigma)$ is said to be **regular generalized α -quotient map** (briefly , **$rg\alpha$ - quotient map**) if f is a $rg\alpha$ - continuous and $f^{-1}(A)$ is an open in (X, τ) implies A is a $rg\alpha$ -open set in (Y, σ) .

Proposition(3-2) :

If a map $f: (X, \tau) \rightarrow (Y, \sigma)$ is surjection , $rg\alpha$ - continuous and $rg\alpha$ -open map , Then f is a $rg\alpha$ -quotient map .

Proof :

From definition (3-1) we only need to prove that if $f^{-1}(A)$ is an open set in (X, τ) implies A is a $rg\alpha$ -open set in (Y, σ) . Let $f^{-1}(A)$ be an open set in (X, τ) . Since f is a $rg\alpha$ -open map

. Thus, $f(f^{-1}(A))$ is a $rg\alpha$ -open set in (Y, σ) and also since f is surjection map. Then $f(f^{-1}(A)) = A$. Hence, A is $rg\alpha$ -open set in (Y, σ) . Therefore, f is a $rg\alpha$ -quotient map.

Proposition(3-3):

If a map $f: (X, \tau) \rightarrow (Y, \sigma)$ is an α -quotient map. Then f is a $rg\alpha$ -quotient map.

Proof:

Let f be an α -quotient map. Then f is surjection, α -continuous map and by using Remark(2-9) step-1- (every α -continuous function is a $rg\alpha$ -continuous), we get f is a $rg\alpha$ -continuous map. Let $f^{-1}(A)$ be an open set in (X, τ) and since f is α -quotient map implies that A is α -open set in (Y, σ) and by using Remark(2-4) step-3- (every α -open set is $rg\alpha$ -open). Thus, A is $rg\alpha$ -open set in (Y, σ) . Therefore, f is a $rg\alpha$ -quotient map.

Corollary(3-4):

Every quotient map is a $rg\alpha$ -quotient map.

Proof:

It follows from Remark(2-10) and proposition (3-3).

The converse of proposition(3-3) and corollary(3-4) are not true in general, as the following example show:

Example(3-1):

Let $X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}\}, Y = \{1, 2\}$ and $\sigma = \{\emptyset, \{1\}\}$.

The map $f: (X, \tau) \rightarrow (Y, \sigma)$ is defined as $f(a) = 2$ and $f(b) = f(c) = 1$. Then f is $rg\alpha$ -quotient map, but f is not quotient (α -quotient map). Since $f^{-1}(\{2\}) = \{a\}$ is an open set in (X, τ) , but $\{2\}$ is not open and α -open set in (Y, σ) .

The following proposition and results give the condition in order to every $rg\alpha$ -closed (resp. $rg\alpha$ -open) set is a closed (resp. open) set.

Proposition(3-5):

If a space (X, τ) is a $T^{*1/2}$ -space. Then every $rg\alpha$ -closed set in X is a closed set.

Proof:

It follows from the Remark (2-4)- step-4- and definition(2-7).

Corollary(3-6):

If a space (X, τ) is a $T^{*1/2}$ -space, Then every $rg\alpha$ -open set in X is an open set.

Proof:

Let A be an $rg\alpha$ -open set in (X, τ) . Then A^c is a $rg\alpha$ -closed set in (X, τ) . Since X is a $T^{*1/2}$ -space. Then A^c is a closed set in X . Hence A is an open set in X .

Next, in the following proposition we give some conditions to make the converse of a proposition (3-3) and corollary(3-4) are true:

Proposition(3-7):

If $f: (X, \tau) \rightarrow (Y, \sigma)$ is $rg\alpha$ -quotient map and X, Y are $T^{*1/2}$ -spaces. Then f is

- (i) α -quotient map.
- (ii) quotient map.

Proof: (i) Let A be an open set in (Y, σ) . Since f is $rg\alpha$ -quotient map. Then f is $rg\alpha$ -continuous surjection map. Thus, $f^{-1}(A)$ is $rg\alpha$ -open set in (X, τ) . Since (X, τ) is $T^{*1/2}$ -space and by using corollary (3-6), we get $f^{-1}(A)$ is an open set in (X, τ) and since (every open set is α -open). Then, $f^{-1}(A)$ is α -open set in (X, τ) . Thus, f is α -continuous map. Now

suppose $f^{-1}(A)$ is an open set in (X, τ) . By hypothesis f is $rg\alpha$ -quotient map. This implies A is $rg\alpha$ -open set in (Y, σ) and also since Y is $T^*_{1/2}$ -space and by corollary (3-6), we have A is an open set in (Y, σ) and by using Remark(2-4) step-1- we obtain A is α -open set in (Y, σ) . Hence, f is α -quotient map.

The proof of step-ii- is similar to step-i-.

Proposition(3-8):

Every \hat{g} -quotient map is a $rg\alpha$ -quotient map.

Proof:

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be \hat{g} -quotient map. Then f is surjection and \hat{g} -continuous map and let A be any open set in (Y, σ) . Thus, $f^{-1}(A)$ is \hat{g} -open set in (X, τ) and using Remark(2-4) step-3- we get, $f^{-1}(A)$ is $rg\alpha$ -open set in (X, τ) . This implies f is $rg\alpha$ -continuous map. Let $f^{-1}(A)$ be an open set in (X, τ) . And also since f is \hat{g} -quotient map implies that A is \hat{g} -open set in (Y, σ) and by using Remark(2-4) step-3- we have, A is $rg\alpha$ -open set in (Y, σ) . Therefore, f is a $rg\alpha$ -quotient.

The following example show that the converse of proposition(3-8) is not true :

Example(3-2): Let $X = \{a, b, c\}, \tau = \{X, \emptyset, \{a\}, \{a, c\}\}, Y = \{a, b\}$ and $\sigma = \{Y, \emptyset, \{a\}\}$. The map $f: (X, \tau) \rightarrow (Y, \sigma)$ is defined as $f(a)=f(c)=b$ and $f(b)=a$. Then f is $rg\alpha$ -quotient map but f is not \hat{g} -quotient map. Since $f^{-1}(\{b\}) = \{a, c\}$ is an open set in (X, τ) but $\{b\}$ is not \hat{g} -open set in (Y, σ) .

In the following proposition we give the condition to make the converse of a proposition (3-8) is true:

Proposition(3-9):

If $f: (X, \tau) \rightarrow (Y, \sigma)$ is $rg\alpha$ -quotient map and X, Y are $T^*_{1/2}$ -spaces. Then f is \hat{g} -quotient map.

Proof:

Let A be an open set in (Y, σ) . Since f is $rg\alpha$ -quotient map. Then f is surjection and $rg\alpha$ -continuous map. This implies that $f^{-1}(A)$ is an $rg\alpha$ -open set in (X, τ) . Since X is $T^*_{1/2}$ -space and by using corollary (3-6), we get $f^{-1}(A)$ is an open set in (X, τ) and by using Remark(2-4) step-1- we get, $f^{-1}(A)$ is \hat{g} -open set in (X, τ) . Thus, f is \hat{g} -continuous map. Now suppose $f^{-1}(A)$ is an open set in (X, τ) . By hypothesis f is $rg\alpha$ -quotient map. This implies A is $rg\alpha$ -open set in (Y, σ) and also since Y is $T^*_{1/2}$ -space and by corollary (3-6), we have A is an open set in (Y, σ) and by using Remark(2-4) step-1- we obtain A is \hat{g} -open set in (Y, σ) . Therefore, f is \hat{g} -quotient map.

Now, we give other type of $rg\alpha$ -quotient map is called strongly $rg\alpha$ -quotient map.

Definition(3-10):

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a surjection map. Then f is said to be **strongly regular generalized α -quotient map** (briefly, **strongly $rg\alpha$ -quotient map**) provided a subset A of (Y, σ) is open in (Y, σ) if and only if $f^{-1}(A)$ is $rg\alpha$ -open in (X, τ) .

Proposition(3-11):

If $f: (X, \tau) \rightarrow (Y, \sigma)$ is a strongly $rg\alpha$ -quotient map, then f is a $rg\alpha$ -quotient.

Proof:

Let f be strongly $rg\alpha$ -quotient map and let A be an open set in (Y, σ) . Then $f^{-1}(A)$ is $rg\alpha$ -open set in (Y, σ) . Thus, f is $rg\alpha$ -continuous. Now let $f^{-1}(A)$ be an open set in (X, τ) . By using remark(2-4) step-3-, we get

$f^{-1}(A)$ is $rg\alpha$ -open set in (X, τ) . Since f is strongly $rg\alpha$ -quotient . This implies that A is an open set in (Y, σ) , also by remark(2-4) step-2- we have A is $rg\alpha$ -open set in (Y, σ) . Therefore, f is $rg\alpha$ -quotient map .

The converse of proposition(3-11) is not true . as see in the following example .

Example(3-3) :

Let $X = \{ a, b, c \}, \tau = \{ X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, Y = \{1, 2\}$ and $\sigma = \{ Y, \emptyset \}$.

The map $f: (X, \tau) \rightarrow (Y, \sigma)$ is defined as $f(a)=1$ and $f(b)=f(c)=2$. Then f is $rg\alpha$ - quotient map . But f is not strongly $rg\alpha$ -quotient map . $f^{-1}(\{1\}) = \{a\}$ is $rg\alpha$ - open set in (X, τ) , but $\{1\}$ is not open set in (Y, σ) .

The following proposition we give the condition to make the converse of a proposition (3-11) is true:

Proposition(3-12):

If $f: (X, \tau) \rightarrow (Y, \sigma)$ is $rg\alpha$ -quotient map and X, Y are $T^*_{1/2}$ -spaces . Then f is strongly $rg\alpha$ -quotient .

Proof:

Let A be an open set in (Y, σ) . Since f is $rg\alpha$ - quotient map . Then f is surjection and $rg\alpha$ - continuous map. Thus, $f^{-1}(A)$ is a $rg\alpha$ - open set in (Y, σ) . **Conversely** , let $f^{-1}(A)$ is $rg\alpha$ - open set in (X, τ) . Since X is $T^*_{1/2}$ -space and by using corollary (3-6) , we get $f^{-1}(A)$ is an open set in (X, τ) and also since f is $rg\alpha$ - quotient map . This implies that A is $rg\alpha$ - open set in (Y, σ) . Since Y is $T^*_{1/2}$ -space , we have A is an open set in (Y, σ) . Therefore , f is strongly $rg\alpha$ -quotient map.

Remark(3-13): The concepts of (quotient maps , α - quotient maps , \hat{g} - quotient maps) are independent to strongly $rg\alpha$ -quotient maps . As, shown in the following examples :

Example(3-4):

Let $X = \{ a, b, c, d \}, \tau = \{ X, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{b, d\}, \{a, b, c\}, \{b, c, d\} \}$, $Y = \{ a, b, c \}$ and $\sigma = \{ Y, \emptyset, \{a\}, \{b\}, \{a, b\} \}$. Define a map $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a)=b$, $f(b)=f(d)= a$ and $f(c)=c$. Then f is strongly $rg\alpha$ -quotient map , but f is not quotient \hat{g} -quotient map and α -quotient map . Since $f^{-1}(\{c\}) = \{c\}$ is an open set in (X, τ) . But $\{c\}$ is not open \hat{g} -open and α -open set in (Y, σ) .

Example(3-5):

Let $X = \{ a, b, c \}, \tau = \{ X, \emptyset, \{a\} \}, Y = \{ a, b \}$ and $\sigma = \{ Y, \emptyset, \{a\} \}$. The map $f: (X, \tau) \rightarrow (Y, \sigma)$ is defined as $f(a)=a$ and $f(b)=f(c)=b$. Then f is quotient and α -quotient map , but f is not strongly $rg\alpha$ -quotient map . Since $f^{-1}(\{b\}) = \{b, c\}$ is not $rg\alpha$ - open set in (X, τ) . But $\{b\}$ is not open set in (Y, σ) .

Next, the following proposition we give the condition to make Remark(3-13) is true:

Proposition(3-14):

If X, Y are two $T^*_{1/2}$ -spaces . Then a map $f: (X, \tau) \rightarrow (Y, \sigma)$ is strongly $rg\alpha$ -quotient map if and only if f is quotient map .

Proof:

Suppose that f is strongly $rg\alpha$ -quotient map . Then by corollary (3-6) , we get f is $rg\alpha$ -quotient map and since X, Y are $T^*_{1/2}$ -spaces and by using proposition(3-5) step-ii- we have f is quotient map . **conversely**, let f is quotient map. To show f is strongly $rg\alpha$ -quotient map . Let A is an open set in (Y, σ) . Since f is quotient map. Thus, $f^{-1}(A)$ is an open set in (X, τ) . By using Remark(2-4) we get $f^{-1}(A)$ is $rg\alpha$ - open set in (X, τ) .Now , let $f^{-1}(A)$ is $rg\alpha$ -open set in (X, τ) . Since X is $T^*_{1/2}$ -space and by using corollary (3-6) , we obtain $f^{-1}(A)$ is an open set in (X, τ) . Since f is quotient map , then f is surjection and A is an open set in (Y, σ) . Therefore , f is strongly $rg\alpha$ -quotient map .

Similarly, we prove the following proposition :

Proposition(3-15):

If X, Y are two $T^*_{1/2}$ -spaces . Then a map $f: (X, \tau) \rightarrow (Y, \sigma)$ is strongly $rg\alpha$ -quotient map if and only if

- (i) f is α - quotient map.
- (ii) f is \hat{g} - quotient map.

Definition(3-16) :

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a surjection map . Then f is said to be **regular generalized α^* -quotient map** (briefly . **$rg\alpha^*$ -quotient map**) if f is a $rg\alpha$ - irresolute and $f^{-1}(A)$ is $rg\alpha$ -open in (X, τ) implies A is an open set in (Y, σ) .

Proposition(3-17) :

Every $rg\alpha^*$ - quotient map is a $rg\alpha$ - irresolute .

Proof : It follows from definition (3-16) .

The converse of proposition(3-17) is not true , as see that in the following example.

Example(3-6) : Let $X = \{ a, b, c, d \}$, $\tau = \{ X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{a, b, c\} \}$, $Y = \{1, 2, 3\}$ and $\sigma = \{ Y, \emptyset, \{1\}, \{2\}, \{1, 2\} \}$. The map $f: (X, \tau) \rightarrow (Y, \sigma)$ is defined as $f(a)=1$, $f(b) = 2$ and $f(c)=f(d)=3$. Then f is $rg\alpha$ - irresolute map . But f is not $rg\alpha^*$ -quotient map . $f^{-1}(\{3\}) = \{c, d\}$ is $rg\alpha$ - open set in (X, τ) , but $\{3\}$ is not open set in (Y, σ) .

Proposition(3-18):

Every $rg\alpha^*$ - quotient map is strongly $rg\alpha$ - quotient map.

Proof:

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be $rg\alpha^*$ -quotient map and let A be an open set in (Y, σ) , by using Remark(2-4) step-3- we get A is a $rg\alpha$ - open set in (Y, σ) . Since f is $rg\alpha^*$ -quotient map .Then f is surjection and $rg\alpha$ - irresolute map .Thus, $f^{-1}(A)$ is $rg\alpha$ - open set in (X, τ) . **Conversely**, If $f^{-1}(A)$ is $rg\alpha$ - open set in (X, τ) . Since f is $rg\alpha^*$ - quotient map . This implies that A is an open set in (Y, σ) . Therefore, f is strongly $rg\alpha$ - quotient map.

Corollary (3-19) :

Every $rg\alpha^*$ - quotient map is $rg\alpha$ - quotient map.

Proof :

This follows from proposition (3-18) and proposition (3-11).

The converse of proposition(3-18) and corollary(3-19) are not true in general , as the following examples show :

Example(3-7):

Let $X=\{a ,b ,c ,d\}$, $\tau= \{ X, \emptyset, \{a\},\{b\},\{c\},\{a ,b\},\{a ,c\},\{b ,c\},\{a ,b ,c\},\{b ,c ,d\} \}$ $Y=\{1,2,3\}$ and $\sigma= \{ Y, \emptyset, \{1\} \}$. The map $f: (X, \tau) \rightarrow (Y, \sigma)$ is defined as $f(a)= 1$, $f(b) = f(c) = 2$ and $f(d)=3$. Then f is strongly $rg\alpha$ -quotient map , but f is not $rg\alpha^*$ - quotient map. Since $f^{-1}(\{2,3\}) = \{b ,c ,d\}$ is $rg\alpha$ - open in (X, τ) , but $\{2,3\}$ is not open in (Y, σ) .

Example(3-8):

Let $X=\{a ,b ,c ,d\}$, $\tau= \{ X, \emptyset, \{a\},\{b\},\{a ,b\},\{a ,b ,c\} \}$, $Y=\{1,2,3\}$ and $\sigma= \{ Y, \emptyset, \{1\} \}$. The map $f: (X, \tau) \rightarrow (Y, \sigma)$ is defined as $f(a)= 1$, $f(b) =3$ and $f(c) =f(d)=2$. Then f is $rg\alpha$ -quotient map , but f is not $rg\alpha^*$ - quotient map . Since $f^{-1}(\{2\}) =\{c ,d\}$ is $rg\alpha$ - open set in (X, τ) , but $\{2\}$ is not open set in (Y, σ) .

The following proposition we give the condition to make the converse of a proposition (3-18) and corollary(3-19) are true:

Proposition(3-20):

If $f: (X, \tau) \rightarrow (Y, \sigma)$ is strongly $rg\alpha$ -quotient map and Y is $T^*_{1/2}$ -space . Then f is $rg\alpha^*$ -quotient map .

Proof:

Let A be $rg\alpha$ -open set in (Y, σ) . Since Y is $T^*_{1/2}$ -space and by using corollary (3-6) , we get A is an open set in (Y, σ) . Also since f is strongly $rg\alpha$ -quotient map . Thus , $f^{-1}(A)$ is $rg\alpha$ - open set in (X, τ) . Hence, f is $rg\alpha$ -irresolute map. Now if $f^{-1}(A)$ is $rg\alpha$ - open set in (X, τ) and since f is strongly $rg\alpha$ -quotient map This implies that A is an open set in (Y, σ) . Therefore, f is $rg\alpha^*$ - quotient map.

Similarly, we prove the following proposition :

Proposition(3-21):

If $f: (X, \tau) \rightarrow (Y, \sigma)$ is $rg\alpha$ -quotient map and X, Y are $T^*_{1/2}$ -spaces . Then f is $rg\alpha^*$ - quotient map .

Remark(3-22):

The concepts of (quotient maps , α - quotient maps , \hat{g} - quotient maps and $rg\alpha^*$ -quotient maps) are independent of each other . As shown in the following examples :

Example(3-9):

Let $X=\{ a ,b ,c ,d \}$, $\tau= \{ X, \emptyset, \{a\}, \{b\},\{a ,b\},\{a ,b ,c\} \}$, $Y=\{a ,,b\}$ and $\sigma= \{ Y, \emptyset, \{a\} \}$ Define a map $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a)= f(b)=f(c)= a$ and $f(d)=b$. Then f is quotient (α -quotient and \hat{g} -quotient) map ,but f is not $rg\alpha^*$ -quotient map. Since $f^{-1}(\{b\}) =\{d\}$ is $rg\alpha$ - open set in (X, τ) , but $\{b\}$ is not open set in (Y, σ) .

Example(3-10):

Let $X = \{a, b, c\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$, $Y = \{1, 2\}$ and $\sigma = \{Y, \emptyset, \{1\}, \{2\}\}$. Define a map $f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = f(b) = 1$ and $f(c) = 2$. Then f is $rg\alpha^*$ -quotient, but f is not quotient (α -quotient) map. Since $\{2\}$ is an open set in (Y, σ) , but $f^{-1}(\{2\}) = \{c\}$ is not open and α -open set in (X, τ) .

Example(3-11):

Let $X = \{a, b, c, d\}$, $\tau = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}, \{b, c\}, \{a, b, c\}, \{a, b, d\}\}$, $Y = \{a, b, c\}$ and $\sigma = \{Y, \emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, \{a, b, c\}, \{b, c, d\}\}$. Define a map

$f: (X, \tau) \rightarrow (Y, \sigma)$ by $f(a) = f(b) = b$, $f(c) = a$ and $f(d) = c$. Then f is $rg\alpha^*$ -quotient, but f is not \hat{g} -quotient map. Since $\{a, c\}$ is an open set in (Y, σ) , but $f^{-1}(\{a, c\}) = \{c, d\}$ is not \hat{g} -open set in (X, τ) .

The following proposition give the condition in order to Remark(3-22) true.

Proposition(3-23):

If X, Y are two $T_{1/2}^*$ -spaces. Then a map $f: (X, \tau) \rightarrow (Y, \sigma)$ is $rg\alpha^*$ -quotient map if and only if

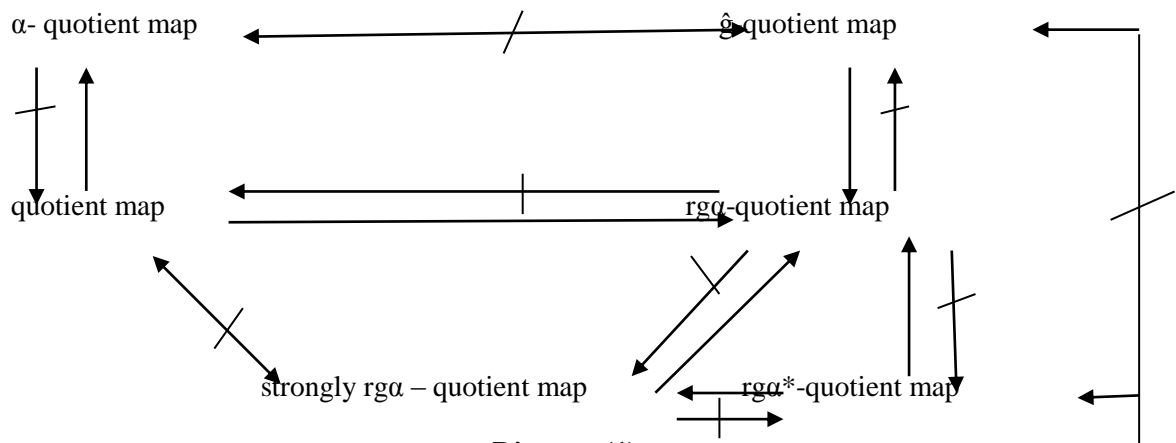
- (i) f is quotient map.
- (ii) f is α -quotient map.
- (iii) f is \hat{g} -quotient map.

Proof:

- (i) Follows from proposition(3-18) and proposition(3-14).
- (ii) Follows from proposition(3-18) and proposition(3-15) step-i.
- (iii) Follows from proposition(3-18) and proposition(3-15) step-ii.

Remark(3-24):

From the above discussion and know results we have the following implications.



Summarized The Relationships Between rga-Quotient Mappings Types

Next, we give some propositions about the composition of rga- quotient mappings types :

Proposition(3-25) :

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be an open surjection rga- irresolute map and

$g: (Y, \sigma) \rightarrow (Z, \mu)$ be a rga-quotient map .Then $gof: (X, \tau) \rightarrow (Z, \mu)$ is rga-quotient map .

Proof:

Let A be any open set in (Z, μ) . Since g is rga-quotient map . Thus, $g^{-1}(A)$ is a rga- open set in (Y, σ) , also since f is rga- irresolute map . Thus , $f^{-1}(g^{-1}(A))$ is a rga- open set in (X, τ) . But $f^{-1}(g^{-1}(A)) = (gof)^{-1}(A)$. So $(gof)^{-1}(A)$ is a rga- open set in (X, τ) . This implies that gof is a rga- continuous maps . Now , suppose that $(gof)^{-1}(A)$ is a rga- open set in (X, τ) for a subset $A \subseteq Z$. Since $f^{-1}(g^{-1}(A)) = (gof)^{-1}(A)$.That is $f^{-1}(g^{-1}(A))$ is an open set in (X, τ) . By hypothesis f is an open surjection map Then , $f(f^{-1}(g^{-1}(A)))$ is an open set in (Y, σ) . It follows that $g^{-1}(A)$ is an open set in (Y, σ) . Since g is rga-quotient map . This implies A is a rga- open set in (Z, μ) . Therefore , $gof: (X, \tau) \rightarrow (Z, \mu)$ is rga-quotient map .

Proposition(3-26):

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be rga*- open surjection rga- irresolute map and

$g: (Y, \sigma) \rightarrow (Z, \mu)$ be strongly rga-quotient map .Then $gof: (X, \tau) \rightarrow (Z, \mu)$ is strongly rga-quotient map .

Proof:

Let A be any open set in (Z, μ) . Since g is strongly rga-quotient map . Thus, $g^{-1}(A)$ is a rga- open set in (Y, σ) , also since f is rga- irresolute map, then, $f^{-1}(g^{-1}(A))$ is a rga- open set in (X, τ) . **Conversely** , suppose that $(gof)^{-1}(A)$ is a rga- open set in (X, τ) for a subset $A \subseteq Z$. Since $f^{-1}(g^{-1}(A)) = (gof)^{-1}(A)$.That is $f^{-1}(g^{-1}(A))$ is rga- open set in (X, τ) , and since f is rga*- open surjection map. Thus $f(f^{-1}(g^{-1}(A)))$ is rga- open set in (Y, σ) . It follows that $g^{-1}(A)$ is rga-open set in (Y, σ) . Since g is strongly rga-quotient map . This implies A is an open set in

(Z, μ) . Therefore , $gof: (X, \tau) \rightarrow (Z, \mu)$ is strongly rga-quotient map .

Proposition(3-27) :

Let $f: (X, \tau) \rightarrow (Y, \sigma)$ be a rga-quotient map and X, Y are $T^*_{1/2}$ -spaces . Then

$g : (Y, \sigma) \rightarrow (Z, \mu)$ is strongly $rg\alpha$ - continuous if and only if $gof : (X, \tau) \rightarrow (Z, \mu)$ is strongly $rg\alpha$ -continuous .

Proof:

Let g is strongly $rg\alpha$ - continuous and A is any a $rg\alpha$ - open set in (Z, μ) . Thus, $g^{-1}(A)$ is an open set in (Y, σ) . Then $f^{-1}(g^{-1}(A))$ is a $rg\alpha$ - open set in (X, τ) . Since X is a $T^*_{1/2}$ -space and by using corollary (3-6) , we get $f^{-1}(g^{-1}(A))$ is an open set in (X, τ) . Hence, the composition $gof : (X, \tau) \rightarrow (Z, \mu)$ is strongly $rg\alpha$ -continuous . **Conversely** , assume that the composition gof is strongly $rg\alpha$ -continuous . Then for any $rg\alpha$ - open set A in Z , $f^{-1}(g^{-1}(A))$ is an open set in (X, τ) . Since f is $rg\alpha$ -quotient map. This implies that $g^{-1}(A)$ is a $rg\alpha$ -open set in (Y, σ) . Also since Y is a $T^*_{1/2}$ -space, we obtain $g^{-1}(A)$ is an open set in (Y, σ) . Therefore , g is strongly $rg\alpha$ - continuous .

Proposition(3-28):

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a surjection $rg\alpha^*$ - open and $rg\alpha$ - irresolute map ,
 $g : (Y, \sigma) \rightarrow (Z, \mu)$ be $rg\alpha^*$ - quotient map. Then $gof : (X, \tau) \rightarrow (Z, \mu)$ is $rg\alpha^*$ -quotient map.

Proof:

Let A be a $rg\alpha$ - open set in (Z, μ) . Since g is a $rg\alpha^*$ - quotient map . Thus , $g^{-1}(A)$ is a $rg\alpha$ - open set in (Y, σ) . Also since f is $rg\alpha$ - irresolute .Then $f^{-1}(g^{-1}(A))$ is a $rg\alpha$ - open set in (X, τ) . Hence, gof is $rg\alpha$ - irresolute . Assume that $(gof)^{-1}(A)$ is a $rg\alpha$ - open set in (X, τ) .for a subset $A \subseteq Z$. Thus , $f^{-1}(g^{-1}(A))$ is a $rg\alpha$ - open set in (X, τ) . Since f is a surjection $rg\alpha^*$ - open, then $f(f^{-1}(g^{-1}(A)))$ is $rg\alpha$ -open set in (Y, σ) . Thus $g^{-1}(A)$ is a $rg\alpha$ -open set in (Y, σ) . Since g is $rg\alpha^*$ - quotient map . This implies that A is an open set in (Z, μ) . Therefore , $gof : (X, \tau) \rightarrow (Z, \mu)$ is $rg\alpha^*$ -quotient map.

Proposition(3-29):

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a strongly $rg\alpha$ - quotient and $rg\alpha$ - irresolute ,
 $g : (Y, \sigma) \rightarrow (Z, \mu)$ is $rg\alpha^*$ - quotient map. Then $gof : (X, \tau) \rightarrow (Z, \mu)$ is $rg\alpha^*$ -quotient map.

Proof:

Let A be a $rg\alpha$ - open set in (Z, μ) . Since g is a $rg\alpha^*$ - quotient map . Thus , $g^{-1}(A)$ is a $rg\alpha$ - open set in (Y, σ) . Also since f is $rg\alpha$ - irresolute . we have $f^{-1}(g^{-1}(A))$ is a $rg\alpha$ - open set in (X, τ) .But $f^{-1}(g^{-1}(A)) = (gof)^{-1}(A)$.That is $(gof)^{-1}(A)$ is a $rg\alpha$ - open set in (X, τ) . Hence, gof is $rg\alpha$ - irresolute . Now assume that $(gof)^{-1}(A)$ is a $rg\alpha$ - open set in (X, τ) for a subset $A \subseteq Z$. Thus , $f^{-1}(g^{-1}(A))$ is a $rg\alpha$ - open set in (X, τ) . Since f is strongly $rg\alpha$ - quotient $g^{-1}(A)$ is an open set in (Y, σ) and by using Remark(2-4) step-ii- we get $g^{-1}(A)$ is a $rg\alpha$ -open set in (Y, σ) . Since g is a $rg\alpha^*$ -quotient map . This implies that A is an open set in (Z, μ) . Therefore , gof is $rg\alpha^*$ -quotient map.

Similarly, we prove the following proposition .

Proposition(3-30):

Let $f : (X, \tau) \rightarrow (Y, \sigma)$ be a strongly $rg\alpha$ - quotient and $rg\alpha$ - irresolute ,
 $g : (Y, \sigma) \rightarrow (Z, \mu)$ be a strongly $rg\alpha$ - quotient .Then $gof : (X, \tau) \rightarrow (Z, \mu)$ is strongly $rg\alpha$ -quotient map .

Proposition(3-31):

If $f : (X, \tau) \rightarrow (Y, \sigma)$ and $g : (Y, \sigma) \rightarrow (Z, \mu)$ are two $rg\alpha^*$ - quotient map. Then $gof : (X, \tau) \rightarrow (Z, \mu)$ is $rg\alpha^*$ -quotient map .

Proof:

Let A be a $rg\alpha$ - open set in (Z, μ) . Since g is a $rg\alpha^*$ - quotient map . Thus , $g^{-1}(A)$ is a $rg\alpha$ - open set in (Y, σ) . Also since f is $rg\alpha^*$ - quotient map. we have $f^{-1}(g^{-1}(A))$ is a $rg\alpha$ - open set in (X, τ) . But $f^{-1}(g^{-1}(A)) = (gof)^{-1}(A)$.That is $(gof)^{-1}(A)$ is a $rg\alpha$ - open set in (X, τ) . Hence, gof is $rg\alpha$ - irresolute . Now assume that $(gof)^{-1}(A)$ is a $rg\alpha$ - open set in (X, τ) for a subset $A \subseteq Z$. But $f^{-1}(g^{-1}(A)) = (gof)^{-1}(A)$.Thus , $f^{-1}(g^{-1}(A))$ is a $rg\alpha$ - open set in (X, τ) . Since f is $rg\alpha^*$ - quotient map , then $g^{-1}(A)$ is an open set in (Y, σ) and by using Remark(2-4) step-3- we get $g^{-1}(A)$ is a $rg\alpha$ -open set in (Y, σ) . Since g is a $rg\alpha^*$ -quotient map . This implies that A is an open set in (Z, μ) . Therefore , gof is $rg\alpha^*$ -quotient map.

Proposition(3- 32):

If $f: (X, \tau) \rightarrow (Y, \sigma)$ is $rg\alpha^*$ - quotient , $g: (Y, \sigma) \rightarrow (Z, \mu)$ is $rg\alpha$ - quotient map and is a $T^*_{1/2}$ -space .Then $gof: (X, \tau) \rightarrow (Z, \mu)$ is .

(i) $rg\alpha^*$ -quotient map.

(ii) strongly $rg\alpha$ -quotient map .

Proof: (i) Let A be an open set in (Z, μ) . Thus , $g^{-1}(A)$ is a $rg\alpha$ - open set in (Y, σ) . since f is $rg\alpha^*$ - quotient ,we have $f^{-1}(g^{-1}(A))$ is a $rg\alpha$ - open set in (X, τ) . But $f^{-1}(g^{-1}(A)) = (gof)^{-1}(A)$.That is $(gof)^{-1}(A)$ is a $rg\alpha$ - open set in (X, τ) . Hence, gof is $rg\alpha$ - irresolute .Now assume that $(gof)^{-1}(A)$ is a $rg\alpha$ - open set in (X, τ) for a subset $A \subseteq Z$. Since $f^{-1}(g^{-1}(A)) = (gof)^{-1}(A)$. Thus $f^{-1}(g^{-1}(A))$ is a $rg\alpha$ - open set in (X, τ) . Since f is $rg\alpha^*$ - quotient map , then $g^{-1}(A)$ is an open set in (Y, σ) , also since g is a $rg\alpha$ - quotient map .This implies that A is a $rg\alpha$ -open set in (Z, μ) .Since Z is $T^*_{1/2}$ -space and by using corollary (3-6) , we have A is an open set in (Z, μ) . Therefore, gof is a $rg\alpha^*$ -quotient map .

The proof of step-ii- is similar to step-i- .

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