Subclass of P-Valent Functions with Negative Coefficients Defined by a Linear Operator with Differential Subordination

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Abstract .

Making use of the linear operator $J_p^m(\lambda,l)$,We introduce and study a subclass $\mathcal{N}_p^m(\lambda,l,\alpha,A,B)$ of analytic and p-valent functions in the open unit disk U. We obtain coefficient inequality, neighborhood property ,extreme points and integral means inequalities.

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1. Introduction

Let $\mathcal{M}_p(n)$ denote the class of functions f of the form:

$$f(z)=z^p+\sum_{k=n+p}^{\infty}a_kz^k\ ,\quad (p,n\in\mathbb{N}=\{1,2,3,\dots\}) \tag{1}$$

which are analytic and p-valent in the open unit disk $U=\{z:z\in\mathbb{C};|z|<1\}$.

Let $\mathcal{N}_p(n)$ be the subclass of $\mathcal{M}_p(n)$, consisting of functions of the form :

$$f(z) = z^p - \sum_{k=n+p}^{\infty} a_k z^k \,, \qquad (a_k \geq 0\,, z \in U) \tag{2}$$

which are analytic and p-valent in the open unit disk U.

A function $f \in \mathcal{N}_p(n)$ is said to be in the class $\mathcal{S}_p^*(n,\tau)$ of p – valent starlike of order τ in U if and only if it satisfies the inequality :

$$Re\left\{\frac{zf'(z)}{f(z)}\right\} > \tau\,, \quad (z \in U; 0 \leq \tau < p). \tag{3}$$

Furthermore, a function $f \in \mathcal{N}_p(n)$ is said to be in the class $\mathcal{C}_p(n,\tau)$ of p – valent convex of order τ in U if and only if it satisfies the inequality :

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \tau, \quad (z \in U; 0 \leq \tau < p). \tag{4}$$

On the other hand, for two functions f and g , and analytic in U , we say that the function f is subordinate to g in U, and write

$$f(z) \prec g(z) \qquad (z \in U),$$

if there exists a Schwarz function $\omega(z)$, analytic in U with

$$\omega(0) = 0 \quad \text{and} \quad |\omega| < 1 \qquad (z \in U),$$

such that

$$f(z) = g(z) \qquad (z \in U).$$

In particular, if the function g is univalent in U, the above subordination is equivalent to

$$f(0) = g(o) \quad \text{and} \quad f(U) \subset g(U).$$

(see ,for details . Duren [1]).

For any function $f \in \mathcal{N}_p(n)$ and $\delta \geq 0$, the (n,δ)-neighborhood of f is defined as,

$$\mathrm{N}_{n,\delta}(f) = \left\{ g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \in \mathcal{N}_p(n) \colon \sum_{k=n+p}^{\infty} k|a_k - b_k| \leq \delta \right\}. \tag{5}$$

In particular, for the function $e(z) = z^p$, we see that ,

$$\mathrm{N}_{n,\delta}(e) = \left\{ g(z) = z^p - \sum_{k=n+p}^{\infty} b_k z^k \in \mathcal{N}_p(n) \colon \sum_{k=n+p}^{\infty} k|b_k| \leq \delta \right\}. \tag{6}$$

The concept of neighborhoods was first introduced by Goodman [2] and then generalized by Ruscheweyh [5].

Definition 1.[4] Prajapat define an generalized multiplier transformation operator $J_p^m(\lambda,l)$ as follows:

$$J_p^{-m}(\lambda,l)f(z) = \frac{p+l}{\lambda} z^{p-(p+l)/\lambda} \int_0^z \xi^{(p+l)\backslash\lambda-p-1} J_p^{-(m-1)}(\lambda,l)f(\xi)d(\xi), (z \in U),$$

$$\vdots$$

$$J_p^{-2}(\lambda,l)f(z) = \frac{p+l}{\lambda} z^{p-(p+l)/\lambda} \int_0^z \xi^{(p+l)\backslash\lambda-p-1} J_p^{-1}(\lambda,l)f(\xi)d(\xi), (z \in U),$$

$$J_p^{-1}(\lambda,l)f(z)=\frac{p+l}{\lambda}z^{p-(p+l)/\lambda}\int_0^z \xi^{(p+l)\backslash\lambda-p-1}f(\xi)d(\xi),(z\in U),$$
$$J_p^0(\lambda,l)f(z)=f(z),$$
$$J_p^1(\lambda,l)f(z)=\frac{\lambda}{p+l}z^{1+p-(p+l)/\lambda}\left(z^{(p+l)\backslash\lambda-p}f(z)\right)',(z\in U),$$
$$J_p^2(\lambda,l)f(z)=\frac{\lambda}{p+l}z^{1+p-(p+l)/\lambda}\left(z^{(p+l)\backslash\lambda-p}J_p^1(\lambda,l)f(z)\right)',(z\in U),$$
$$\vdots$$
$$J_p^m(\lambda,l)f(z)=\frac{\lambda}{p+l}z^{1+p-(p+l)/\lambda}\left(z^{(p+l)\backslash\lambda-p}J_p^{m-1}(\lambda,l)f(z)\right)',(z\in U). \qquad (7)$$

We see that for $f\in\mathcal{N}_p(n)$ is given by (2), we have

$$J_p^m(\lambda,l)f(z)=z^p-\sum_{k=n+p}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m a_k z^k, \qquad (z\in U) \qquad (8)$$

where $\lambda\geq 0, l>-p, p\in\mathbb{N}, m\in\mathbb{Z}=\{0,\pm1,\pm2,\cdots\}$and $z\in U$.

In this paper, we will use operator $J_p^m(\lambda,l)$ and mentioned principle of subordination between analytic functions to define a subclass of $\mathcal{N}_p(n)$ as follows:

Definition 2. For $\lambda\geq 0, l>-p, p\in\mathbb{N}, m\in\mathbb{Z}$ and for the parameters α, A and B such that $-1\leq B<A\leq 1, 0\leq\alpha<p$, we say that a function $f\in\mathcal{N}_p(n)$ is in the class $\mathcal{N}_p^m(\lambda,l,\alpha,\gamma,A,B)$ if it satisfies the following subordination:

$$\frac{1}{p-\alpha}\left(1+\frac{z\left(J_p^m(\lambda,l)f(z)\right)''}{\left(J_p^m(\lambda,l)f(z)\right)'}-\alpha\right)\prec\frac{1+\gamma z}{1+Bz}, \qquad (z\in U) \qquad (9)$$

where $\gamma=(1-\beta)A+\beta B$,$0\leq\beta<1$ and $J_p^m(\lambda,l)f$ given by (8).

2.Main Results

Theorem 1.

A function $f\in\mathcal{N}_p(n)$ be defined by (2). Then $f\in\mathcal{N}_p^m(\lambda,l,\alpha,\gamma,A,B)$ if and only if

$$\sum_{k=n+p}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m[(k-p)(1+B)-(\gamma-B)(p-\alpha)]ka_k\leq p(\gamma-B)(p-\alpha), \quad (10)$$

for $p,n\in\mathbb{N}, \lambda\geq 0, l>-p, m\in\mathbb{Z}$ and $-1\leq B<A\leq 1, 0\leq\alpha<p$.

Proof. Let $f\in\mathcal{N}_p^m(\lambda,l,\alpha,\gamma,A,B)$. Then ,

$$\frac{1}{p-\alpha}\left(1+\frac{z\left(J_p^m(\lambda,l)f(z)\right)''}{\left(J_p^m(\lambda,l)f(z)\right)'}-\alpha\right)\prec\frac{1+\gamma z}{1+Bz}. \qquad (z\in U) \qquad (11)$$

Therefore ,there exists an analytic function ω such that

$$\omega(z)=\frac{\left(1+\frac{z\left(J_p^m(\lambda,l)f(z)\right)''}{\left(J_p^m(\lambda,l)f(z)\right)'}-p\right)}{B\left(1+\frac{z\left(J_p^m(\lambda,l)f(z)\right)''}{\left(J_p^m(\lambda,l)f(z)\right)'}\right)-\left(pB+(\gamma-B)(p-\alpha)\right)}. \qquad (12)$$

Hence,

$$|\omega(z)|=\left|\frac{\left(1+\frac{z\left(J_p^m(\lambda,l)f(z)\right)''}{\left(J_p^m(\lambda,l)f(z)\right)'}-p\right)}{B\left(1+\frac{z\left(J_p^m(\lambda,l)f(z)\right)''}{\left(J_p^m(\lambda,l)f(z)\right)'}\right)-\left(pB+(\gamma-B)(p-\alpha)\right)}\right|$$

$$=\left|\frac{-\sum_{k=n+p}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k(k-p)a_k z^{k-1}}{-p(\gamma-B)(p-\alpha)-\sum_{k=n+p}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[B(k-p)-(\gamma-B)(p-\alpha)]a_k z^{k-1}}\right|$$

$$<1.$$

Thus,

$$Re\left\{\frac{-\sum_{k=n+p}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k(k-p)a_k z^{k-1}}{-p(\gamma-B)(p-\alpha)z^{p-1}-\sum_{k=n+p}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[B(k-p)-(\gamma-B)(p-\alpha)]a_k z^{k-1}}\right\}$$

$$<1. \qquad (13)$$

Taking $|z|=r$, for sufficiently small r with $0<r<1$, the denominator of (13) is positive for all r with $0<r<1$, since $\omega(z)$ is analytic for $|z|<1$. Then , the inequality (13) yields

$$\sum_{k=n+p}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k(k-p)a_k r^{k-1}$$

$$<p(\gamma-B)(p-\alpha)r^{p-1}-\sum_{k=n+p}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[B(k-p)-(\gamma-B)(p-\alpha)]a_k r^{k-1}.$$

Equivalently ,

$$\sum_{k=n+p}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^{m}[(k-p)(1+B)-(\gamma-B)(p-\alpha)]ka_k r^{k-1}$$

$$\leq p(\gamma-B)(p-\alpha)r^{p-1},$$

and (10) follows upon letting $r \to 1$.

Conversely , for $|z| = r, 0 < r < 1$, we have $r^k < r^p$. That is,

$$\sum_{k=n+p}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^{m}[(k-p)(1+B)-(\gamma-B)(p-\alpha)]ka_k r^{k-1}$$

$$\leq \sum_{k=n+p}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^{m}[(k-p)(1+B)-(\gamma-B)(p-\alpha)]ka_k r^{p-1}$$

$$\leq p(\gamma-B)(p-\alpha)r^{p-1}.$$

From (10) ,we have

$$\left|\sum_{k=n+p}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^{m}k(k-p)a_k z^{k-1}\right|$$

$$\leq \sum_{k=n+p}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^{m}k(k-p)a_k r^{k-1}$$

$$< p(\gamma-B)(p-\alpha)r^{p-1} - \sum_{k=n+p}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^{m}k[B(k-p)-(\gamma-B)(p-\alpha)]a_k r^{k-1}.$$

$$< \left|p(\gamma-B)(p-\alpha)r^{p-1} + \sum_{k=n+p}^{\infty}\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^{m}k[B(k-p)-(\gamma-B)(p-\alpha)]a_k r^{k-1}\right|.$$

This proves that

$$\frac{1}{p-\alpha}\left(1+\frac{z\left(J_p^m(\lambda,l)f(z)\right)''}{\left(J_p^m(\lambda,l)f(z)\right)'}-\alpha\right) \prec \frac{1+\gamma z}{1+Bz}, \quad (z \in U)$$

and hence $f \in \mathcal{N}_p^m(\lambda,l,\alpha,\gamma,A,B)$.

Corollary 1. Let the function $f \in \mathcal{N}_p(n)$ be given by (2).If $f \in \mathcal{N}_p^m(\lambda, l, \alpha, \gamma, A, B)$, then

$$a_k \leq \frac{p(\gamma - B)(p - \alpha)}{\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^m k[(k - p)(1 + B) - (\gamma - B)(p - \alpha)]} . \quad (k, p \in \mathbb{N}) \quad (14)$$

The result is sharp for the function given by

$$f(z) = z^p - \sum_{k=n+p}^{\infty} \frac{p(\gamma - B)(p - \alpha)}{\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^m k[(k - p)(1 + B) - (\gamma - B)(p - \alpha)]a_k} z^k. \quad (15)$$

3. Neighborhoods for The class $\boldsymbol{\mathcal{N}_p^m(\lambda, l, \alpha, \gamma, A, B)}$

We determine the neighborhood properties for functions belonging to the subclass $\mathcal{N}_p^m(\lambda, l, \alpha, \gamma, A, B)$.

For $0 \leq \sigma < p$ and $z \in U$, a function $f \in \mathcal{N}_p^m$ is said to be in the class $\mathcal{N}_p^{\sigma,m}(\lambda, l, \alpha, \gamma, A, B)$ if there exists a function $g \in \mathcal{N}_p^m(\lambda, l, \alpha, \gamma, A, B)$ such that

$$\left|\frac{f(z)}{g(z)} - 1\right| < p - \sigma.$$

Theorem 2. If $g \in \mathcal{N}_p^m(\lambda, l, \alpha, \gamma, A, B)$ and

$$\sigma = p - \frac{\delta\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^m [n(1 + B) - (\gamma - B)(p - \alpha)]}{\left(\frac{p + l + \lambda(k - p)}{p + l}\right)^m \left[(n + p)\big(n(1 + B) - (\gamma - B)(p - \alpha)\big)\right] - p(\gamma - B)(p - \alpha)}, (16)$$

then $\mathrm{N}_{n,\delta}(g) \subset \mathcal{N}_p^{\sigma,m}(\lambda, l, \alpha, \gamma, A, B)$.

Proof. Let $f \in \mathrm{N}_{n,\delta}(g)$ .We want to find from (5) that

$$\sum_{k=n+p}^{\infty} k|a_k - b_k| \leq \delta,$$

which yields the coefficient inequality ,

$$\sum_{k=n+p}^{\infty} |a_k - b_k| \leq \frac{\delta}{n + p}, \quad (n, p \in \mathbb{N}). \quad (17)$$

Since , $g \in \mathcal{N}_p^{\sigma,m}(\lambda, l, \alpha, \gamma, A, B)$,we have from Theorem 1.

$$\sum_{k=n+p}^{\infty} b_k \leq \frac{p(\gamma-B)(p-\alpha)}{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m [(k-p)(1+B)-(\gamma-B)(p-\alpha)]n+p}. \tag{18}$$

So that

$$\left|\frac{f(z)}{g(z)}-1\right| < \frac{\sum_{k=n+p}^{\infty} |a_k-b_k|}{1-\sum_{k=n+p}^{\infty} b_k}$$

$$\leq \frac{\delta\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m [n(1+B)-(\gamma-B)(p-\alpha)]}{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m \left[(n+p)\big(n(1+B)-(\gamma-B)(p-\alpha)\big)\right]-p(\gamma-B)(p-\alpha)} = p-\sigma.$$

Thus, by Definition , $f \in \mathcal{N}_p^m(\lambda,l,\alpha,\gamma,A,B)$ for σ given by (16).

This completes the proof.

Theorem 3. If

$$\varphi = \frac{p(\gamma-B)(p-\alpha)}{\left(\frac{p+l+n}{p+l}\right)^m [n(1+B)-(\gamma-B)(p-\alpha)]}, \tag{19}$$

Then $\mathcal{N}_p^m(\lambda,l,\alpha,\gamma,A,B) \subset \mathbb{N}_{n,\delta}(e)$.

Proof. It follows from (10), that if $f \in \mathcal{N}_p^m(\lambda,l,\alpha,\gamma,A,B)$, then

$$\left(\frac{p+l+\lambda n}{p+l}\right)^m [n(1+B)-(\gamma-B)(p-\alpha)] \sum_{k=n+p}^{\infty} ka_k \leq p(\gamma-B)(p-\alpha), \tag{20}$$

Which implies ,

$$\sum_{k=n+p}^{\infty} ka_k \leq \frac{p(\gamma-B)(p-\alpha)}{\left(\frac{p+l+\lambda n}{p+l}\right)^m [n(1+B)-(\gamma-B)(p-\alpha)]} = \varphi. \tag{21}$$

Using (6), we get result.

4. Extreme point for the function class $\mathcal{N}_p^m(\lambda,l,\alpha,\gamma,A,B)$.

Theorem 3. Let

$$f_1(z) = z^p \tag{22}$$

and

$$f_k(z) = z^p - \frac{p(\gamma - B)(p-\alpha)}{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]} z^k. \quad (23)$$

Then , $f \in \mathcal{N}_p^m(\lambda, l, \alpha, \gamma, A, B)$ if and only if can be expressed in the following from :

$$f(z) = \sum_{k=n+p}^{\infty} \Upsilon_k f_k(z),$$

where $\Upsilon \geq 0$ and

$$\sum_{k=n+p-1}^{\infty} \Upsilon_k f_k(z) = 1.$$

Proof. Suppose that

$$f(z) = \sum_{k=n+p}^{\infty} \Upsilon_k f_k(z)$$

$$= z^p - \sum_{k=n+p}^{\infty} \Upsilon_k \frac{p(\gamma - B)(p-\alpha)}{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]} z^k.$$

Then , from Theorem 1 , we have

$$\sum_{k=n+p}^{\infty} \left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]$$

$$\times \frac{p(\gamma - B)(p-\alpha)}{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]} \Upsilon_k$$

$$= p(\gamma - B)(p-\alpha) \sum_{k=n+p}^{\infty} \Upsilon_k = p(\gamma - B)(p-\alpha)\big(1 - \Upsilon_{n+p-1}\big) \leq p(\gamma - B)(p-\alpha).$$

Thus , in view of Theorem 1 .we find that $f \in \mathcal{N}_p^m(\lambda, l, \alpha, \gamma, A, B)$.

Conversely, let us suppose that $f \in \mathcal{N}_p^m(\lambda, l, \alpha, \gamma, A, B)$.Then ,since

$$a_k \leq \frac{p(\gamma - B)(p-\alpha)}{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]} , \qquad (k \geq n+p)$$

we may set

$$\Upsilon_k = \frac{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]}{p(\gamma-B)(p-\alpha)} a_k, (k \geq n+p)$$

and

$$\Upsilon_{n+p-1} = 1 - \sum_{k=n+p}^{\infty} \Upsilon_k.$$

Thus, clearly we have

$$f(z) = z^p - \sum_{k=n+p}^{\infty} \Upsilon_k f_k(z)$$

.

This completes the proof of Theorem 3.

Corollary 2. The extreme points of the function class $\mathcal{N}_p^m(\lambda, l, \alpha, \gamma, A, B)$are given by (22) and (23).

5. Integral means inequalities for the function class $\mathcal{N}_p^m(\lambda, l, \alpha, \gamma, A, B)$

In the year 1925, Littlewood's [3] proved the following subordination theorem.

Theorem 4. If the functions f and g are analytic in U with $f(z) \prec g(z), (z \in U)$, Then for $\mu > 0$ and $z = re^{i\vartheta}$ $(0 < r < 1)$,

$$\int_0^{2\pi} \left|f(re^{i\vartheta})\right|^{\mu} d\vartheta \leq \int_0^{2\pi} \left|g(re^{i\vartheta})\right|^{\mu} d\vartheta \quad .$$

We now make use of Theorem (4) to prove Theorem (5).

Theorem 5. Let $f \in \mathcal{N}_p^m(\lambda, l, \alpha, \gamma, A, B)$. Suppose also that $f_k(z)$ is defined be equation (23). If there exists an analytic function $\omega(z)$ given by

$$[\omega(z)]^{k-p} = \frac{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]}{p(\gamma-B)(p-\alpha)} \sum_{k=n+p}^{\infty} a_k z^{k-p},$$

Therefor $z = re^{i\vartheta}$ and $(0 < r < 1)$,

$$\int_0^{2\pi} \left|f(re^{i\vartheta})\right|^{\mu} d\vartheta \leq \int_0^{2\pi} \left|f_k(re^{i\vartheta})\right|^{\mu} d\vartheta, \quad (\mu > 0) \; .$$

Proof. We must show that

$$\int_0^{2\pi}\left|1-\sum_{k=n+p}^{\infty}a_k z^{k-p}\right|^{\mu}d\vartheta$$

$$\leq \int_0^{2\pi}\left|1-\frac{p(\gamma-B)(p-\alpha)}{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]}z^{k-p}\right|^{\mu}d\vartheta.$$

By applying Littlewood's subordination theorem, it would suffice to show that

$$1-\sum_{k=n+p}^{\infty}a_k z^{k-p} \prec 1-\frac{p(\gamma-B)(p-\alpha)}{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]}z^{k-p}.$$

By setting

$$1-\sum_{k=n+p}^{\infty}a_k z^{k-p} \prec 1-\frac{p(\gamma-B)(p-\alpha)}{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]}[\omega(z)]^{k-p},$$

we find that

$$[\omega(z)]^{k-p}=\frac{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]}{p(\gamma-B)(p-\alpha)}\sum_{k=n+p}^{\infty}a_k z^{k-p},$$

which readily yields $\omega(0)=0$.

Furthermore , by using equation (10), we obtain

$$\left|[\omega(z)]^{k-p}\right|=\left|\frac{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]}{p(\gamma-B)(p-\alpha)}\sum_{k=n+p}^{\infty}a_k z^{k-p}\right|$$

$$\leq \frac{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^m k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]}{p(\gamma-B)(p-\alpha)}\sum_{k=n+p}^{\infty}a_k |z|^{k-p}$$

$$\leq |z|^{k-p}<1.$$

This completes the proof of Theorem(5).

Theorem 6. Let $\mu>0$.If $f\in \mathcal{N}_p^m(\lambda,l,\alpha,\gamma,A,B)$ and $f_k(z)$ is given by (23). Then $z=re^{i\vartheta}$ and $(0<r<1)$,

$$\int_0^{2\pi}\left|f'(re^{i\vartheta})\right|^{\mu}d\vartheta\leq\int_0^{2\pi}\left|f_k{}'(re^{i\vartheta})\right|^{\mu}d\vartheta\,,$$

Proof. It is sufficient to show that

$$1-\sum_{k=n+p}^{\infty}\left(\frac{k}{p}\right)a_k z^{k-p}$$

$$\prec 1-\frac{p(\gamma-B)(p-\alpha)}{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^{m}k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]}z^{k-p}.$$

This follows because

$$|[\omega(z)]|^{k-p}=\left|\sum_{k=n+p}^{\infty}\frac{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^{m}k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]}{p(\gamma-B)(p-\alpha)}a_k z^{k-p}\right|$$

$$\leq |z|^{k-p}\sum_{k=n+p}^{\infty}a_k\frac{\left(\frac{p+l+\lambda(k-p)}{p+l}\right)^{m}k[(k-p)(1+B)-(\gamma-B)(p-\alpha)]}{p(\gamma-B)(p-\alpha)}$$

$$\leq |z|^{k-p}\leq |z|.$$

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