$\mathbf{Weighted(k,n)-arcs\ of\ Type(n-11,n)inPG(2,11)}$

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Abstract

In this paper, we discussed a weighted (k,n) −arcs of type $(n-11,n)$ in projective plane of order eleven. We found a new methodology for the construction of the weighted (k,n) −arcs of two types. We proved that there exist a $(78,n;f)$ −arcs of type $(n-11,n)$ when the points of weight zero is fifty five and the points of weight one form an envelope. Finally, we proved that there is a weighted (k,n) −arcs of type $(n-11,n)$ when the points of weight one lies on only one n −weighting line of a weighted (k,n) −arcs and the points of weight two lies on exactly two n −weighting lines.

Key words: Projective plane of order 11, Weighted (k,n) −arcs of two types.

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1. Introduction

The notion of weighted $(k,n,f)-$ arcs were proposed by Scafati (1971), Barnabei (1979) studied these types of arcs and obtained some particular results about the existence and non-existence of these arcs in $PG(2,q)$ by using a computer programmer. D'Agostini (1979) studied caps with weighted points in$PG(2,q)$, some relations between parameters of$(k,n\ f)$ −caps and characters were found. In particular, D'Agostini (1980) studied a weighted (k,n) −arcs of type $(n-2,n)$ in $PG(2,q)$. Wilson (1986) proved that there is$(88,14,f)$ −arc of type $(11,14)$ in the Galois plane of order 9. Also he proved that there is $(10,7,f)$ −arc of type $(4,7)$ in $PG(2,3)$. Hameed (1989) studied the existence and non-existence weighted (k,n) −arcs in $PG(2,9)$ and he proved that there exist $(81,12,f)$ −arc of type $(9,12)$ and $(85,13,f)$ −arc of type $(10,13)$. Mahmood (1990) discussed $(k,n,f)-$ arcs in $PG(2,5)$. She proved there is $(21,11,f)$ −arc of type $(6,11)$, $(20,10,f)$ −arc of type $(5,10)$ and $(15,8,f)$ −arc of type $(3,8)$. The extensions work of the weighted arcs in $PG(2,9)$ was investigated by Abass (2011). He proved there exist$(81,12,f)$ −arc of type $(9,12)$ and $(76,11,f)$ −arc of type $(8,11)$. Abudljebar (2012), studied the generalization of $(k,n,f)-$ arcs of type $(1,\mathrm{n})$ in $PG(2,q)$ and weighted(k,n) −arcs of type $(\mathrm{n}-4,\mathrm{n})$ in $PG(2,8)$.

Most of the authors were used the table of lines and coordinates of points, to construct the example of a weighted$(k\ ,n)$ −arcs in $PG(2,q)$, while we used a new technique depending on the dual of k-arcs and the envelope in projective plane of order eleven.

2. Preliminaries

Definition 2.1 [Hirschfeld**].** Let $GF(p)=\mathbb{Z}/p\mathbb{Z}, p$ is prime and let $f(x)$ be irreducible polynomial of degree h over $GF(p)$, then

$$GF(q)=GF(p^h)=\frac{GF(p)[x]}{f(x)}=\{a_0+a_1t+\cdots+a_{h-1}t^{h-1}:a_i in\ GF(P), f(t)=0\}$$

Definition 2.2 [Hirschfeld**].** A projective plane over $GF(q)$ is 2-dimentional projective space and denoted by $PG(2,q)$ or π which contains q^2+q+1 lines, every line contains $q+1$ points and satisfy the following axioms:

(i) Any two distinct points determine a unique line;

(ii) Any two distinct lines intersect in exactly one point;

(iii) There exist four distinct points such that no three of them are collinear.

Definition 2.3 [Hirschfeld**].** A t_n-arc is a set of t_n points such that no three points are collinear.

Lemma 2.4 [Hirschfeld**].** Let $t(p)$ be the number of tangents through p of t_n −arc, T_i be the number of i −secants of t_n in $PG(2,q)$, then

(i) $t(p)=q+2-t_n$;

(ii) $T_2=\frac{t_n(t_n-1)}{2}$;

(iii) $T_1=t_nt\ ,\ t=q+2-t_n$;

(iv) $T_0=\frac{q(q-1)}{2}+\frac{t(t-1)}{2}$;

(v) $T_0+T_1+T_2=q^2+q+1$.

Definition 2.5[Hirschfeld**].** The set of t_n lines such that no three are concurrent is called a dual of t_n-arc.

Definition 2.6 [Hirschfeld**].** The set of $(q+1)$ lines such that no three of them are concurrent is called an envelope and which is represent a dual of the conic in $PG(2,q)$, when q is odd.

Lemma 2.7. Let $t(l)$be the number of points lies on l and let S_i be the number of points which pass through it i 2-secant, then

(i) $t(l)=q+2-t_n$;

(ii) $S_2=\frac{t_n(t_n-1)}{2}$;

(iii) $S_1 = t_n t$, $t = q + 2 - t_n$;

(iv) $S_0 = \frac{q(q-1)}{2} + \frac{t(t-1)}{2}$;

(v) $S_0 + S_1 + S_2 = q^2 + q + 1$.

Definition 2.8 [Hirschfeld**].** A (k,n) −arc $\mathcal{H}$ is a set of k points such that there are n but no $n+1$ them are collinear.

Lemma 2.9 [Wilson**]**. For the (k,n)-arc $\mathcal{H}$, the following equations are hold:

(i) $\sum_{i=0}^{n} \tau_i = q^2 + q + 1$;

(ii) $\sum_{i=1}^{n} i\tau_i = k(q+1)$;

(iii)$\sum_{i=2}^{n} \frac{i(i-1)}{2} \tau_i = \frac{k(k-1)}{2}$,

where τ_i are the number of i-secants of(k,n)-arc such that $\mathcal{H} \cap \tau = i$.

Definition 2.10 [D'Agostini**]**. Let π be the projective plane of order q and denoted by p and R are respectively the sets of points and lines of π. Let f be a function from P into the set N of non-negative integers and call the weight of $p \epsilon P$ the value $f(p)$ and the support of $f$the set of points of the plane have non −zero weight. By using f we can define a function $F: R \longrightarrow N$ such that for any $r \epsilon R, F(r) = \sum_{p \epsilon r} f(p)$. We call $F(r)$ the weight of the line r.

Definition 2.11 [D'Agostini**].** A $(k,n;f)$-arc of the plane π is a subset K of the points of the plane such that

(i) K is the support of f ;

(ii) $k = |K|$;

(iii) $n = max\{F(r): r \epsilon R\}$.

Denote ω =$max_{p \epsilon P} f(p)$, V_i^j to the number of lines of weight i through a point of weight $j$and $W = \sum_{j=0}^{\omega} \mathcal{H}_j = \sum_{p \epsilon P} f(p)$. For a $(k,n;f) -$ arc, we have the following important Lemma:

Lemma 2.12 [Hameed**].** For the weighted (k,n) −arcs in $PG(2,11)$, the following statements are holds:

(i) $\omega \leq 11$;

(ii) If p is any point of the plane, then $\sum_{r \epsilon [p]} F(r) = W + qf(p)$, where$[p]$ denote the set of lines through p;

(iii) The weight W of a weighted$(k\,,n)$ −arc satisfies $(n-11)(q+1) \leq W \leq (n-\omega)q + n$;

(iv) Let K be a weighted$(k\,,n)$ −arc of type $(n-11, n), n-11 > 0$ and let p be a point having weight s, then V_m^s and V_n^s are determined p and are given by:

$$V_{n-11}^{s} = \frac{q(n-s) - W + n}{11}$$

and

$$V_n^s = \frac{q(s-n+11)+W-n+11}{11} \;;$$

(v) $q \equiv 0 mod(11)$;

(vi) $k = \sum_{j=1}^{2} l_j$;

(vii) The characters of a weighted $(k\ ,n) -$ arcs K of type $(n-11, n)$ are given by

$$t_{n-11} = [\frac{q+1}{11}][\frac{n(q^2+q+1)}{q+1} - W]$$

and

$$t_n = [\frac{q+1}{11}][W - \frac{(n-11)(q^2+q+1)}{q+1}]$$

Corollary 2.13 [Hameed**].** If $W = (n-11)(q+1)$, then a weighted$(k\ ,n) -$arc is minimal and if $W = (n-\omega)+n$, then a weighted $(k\ ,n) -$arc is maximal.

Principle of Duality 2.14 [Hirschfeld**].** For any space $S = PG(n,q)$, there is a dual space S^*, whose points and primes are respectively primes and points of S. For any theorem true in □, there is an equivalent theorem true in $□^*$.

3. weighted $(□\ ,□)$ **−arcsof type**$(□ - □□, □)$

For a weighted$(□\ ,□) -$arcs of type $(□ - 11, □)$, it is necessary that $□ \geq 11$.If $□ = 11$ we have to consider a $(□, 11) -$arc having only $0 -$weighting line and $11 -$weighting lines, this arc is maximal.

Lemma 3.1. The existence of a weighted $(□\ ,□) -$arcs of type $(□ - 11, □)$ in $□□(2, □)$ with $□ + 1 < □ < 2□ + 2$ requires $□ \equiv 0□□□(11)$

Proof. Directly, from Lemma 2.12 case (v). ■

Lemma 3.2 [D'Agostini**].** The existence of a weighted $(□\ ,□) -$arcs of type $(□ - 11, □)$ in $□□(2, □)$ with $□ + 1 < □ < 2□ + 2$ requires $□_□ = 0, □ \geq 3$.

We used Lemma 2.12 case (iii) to get

$$(□ - 11)(□ + 1) \leq □ \leq (□ - 11)(□ + 1) + 11$$

Lemma 3. 3. For a weighted $(□\ ,□) -$arcs of type $(□ - 11, □)$ in $□□(2, □)$ with □ minimal $(□ = (□ + 1)(□ - 11))$we have

$$□^0_{□-11} = □ + 1, \quad □^1_{□-11} = \frac{10□ + 11}{11}, \quad □^2_{□-11} = \frac{9□ + 11}{11}$$

$$□^0_□ = 0, \quad □^1_□ = \frac{□}{11}, \quad □^2_□ = \frac{2□}{11}$$

Proof. From Lemma 2. 12case (iv), by substituting $□ - 11 = □ + 1$ for Im$□ = \{0, 1, 2\}$.■

Corollary 3.4. There is no points of weight 0 on □ − weighting lines of a weighted (□ , □) −arcs of type ($□ - 11$, □).

For the case $□_0 > 0, □_1 > 0, □_2 > 0$ and $□_□ = 0$ where $3 \leq □ \leq 11$, we have the weight of the points of the (□, □; □) −arc is $\omega = 2$, and by using the minimal case ($□ = (□ - 11)(□ + 1)$) and by the counting the number of lines of □□(2, □) we find the following:

$$□_□ + □_{□-11} = □^2 + □ + 1$$

By counting the number of □ − weighting lines $□_□$ and ($□ - 11$)- weighting lines $□_{□-11}$, and counting the total incidence, it follows that

$$□□_□ + (□ - 11)□_{□-11} = □(□ + 1) = (□ - 11)(□ + 1)^2$$

Consequently, we get

$$□_□ = \frac{(□-11)□}{11} \tag{3.1}$$

$$□_{□-11} = \frac{11□^2 + (22-□)□ + 11}{11} \tag{3.2}$$

Lemma 3.5. The □ − weighting lines of a weighted (□ , □) −arcs of type ($□ - 11$, □) form a dual of $□_□$ −arc in □□(2,11).

Proof. From Lemma 3.3, we have $□_□^2 = 2$, this mean that there are no three □ − weighting lines are concurrent. Then the number of □ − weighting lines $□_□$ form a dual of $□_□$ −arc. ■

Suppose that on □ −weighting lines there are α points of weight 1 and β points of weight 2. Then counting the points of □ −weighting lines, it follows that:

$$\alpha + \beta = □ + 1$$

And counting the weight of points on □ −weighting lines, we have

$$\alpha + 2□ = □$$

Solving these two equations, we obtain

$$\alpha = 2(□ + 1) - □ \tag{3.3}$$

$$\beta = □ - (□ + 1) \tag{3.4}$$

counting the incidences between the points of weight 2 and □ −weighting lines, we get

$$□_2 □_□^2 = □_□ □$$

Making use of Lemma 3.3, equation (3.1) and equation (3.4) we obtain

$$□_2 = \frac{(□-11)(□-□-1)}{2} \tag{3.5}$$

Similarly, counting the incidences between the points of weight *1* and □ −weighting lines, we have

$$□_1 □^1_□ = □_□ □$$

Hence, by using Lemma 3.3, equation (3.2) and equation (3.3), we get

$$□_1 = (□ - 11)(2□ + 2 - □) \tag{3.6}$$

From equations (3.5) and (3.6), counting the points in the plane

$$□_0 + □_1 + □_2 = □^2 + □ + 1$$

$$□_0 = □^2 + □ + 1 - (□ - 11)(2□ + 2 - □) - \frac{(□ - 11)(□ - □ - 1)}{2}$$

Hence

$$2□^2 + (35 - 3□)□ + □^2 - 14□ + 35 - 2□_0 = 0 \tag{3.7}$$

The solution of equation (3.7) exists with respect to □ *i*f $(35 - 3□)^2 - 8(□^2 - 14□ + 35 - 2□_0)$ is square, then

$$(□ - 49)^2 - (1456 - 16□_0) = □□□□□□ \tag{3.8}$$

We discuss the $(□, □; □) -$ arcs of type $(□ - 11, □)$ in $□□(2,11)$ when the points of weight 0 equal to 55. For the value of $□_0 = 55$, the equation (3.7) becomes

$$2□^2 + (35 - 3□)□ + □^2 - 14□ - 75 = 0 \tag{3.9}$$

From the equation (3.8), we conclude that$((□ - 49)^2 - 576)$ must be square.

For the value of $□ = 11$, the equation (3.9) becomes

$$□^2 - 47□ + 552 = 0$$

We have two solutions for □ being non-negative with $□ = 0□□□(□ - (□ - 11))$ which are $□ = 24$ or $□ = 23$.

For *n*= *24*, the weight of the lines either $2□ + 2$ or zero and this gives underling $(□ , □) -$arcs in $□□(2, □)$

4. $(□□, □□; □)$ −arcs of Type $(□□, □□)$ in $□□(□, □□)$

Lemma 4.1. For a $(78, 23;\ □) -$arc of type $(12, 23)$ in $□□(2, 11)$ for the minimal $□ = 144$, we have

$$□^0_{12} = 12 \quad ,□^1_{12} = 11 \quad ,□^2_{12} = 10 \quad ,□^0_{23} = 0 \quad ,□^1_{23} = 1 \quad ,□^2_{23} = 2$$

Proof. From Lemma 3.3 directly, we get the requirements by putting $□ = 23$, $□ = 11$. □

Lemma 4.2. For the existence of a minimal$(78, 23;\ □) -$arc of type $(12, 23)$ with $□_0 = 55$ in $□□(2, 11)$ we have:

(i) The number of 23 −weighting lines $(□_{23})$ is *12*;

(ii) The number of 12 −weighting lines($□_{12}$) is *121*;

(iii) The number of points of weight *2* $(□_2)$ is *66*;

(iv) The number of points of weight *1* $(□_1)$ is *12*.

Proof. From equations (3.1), (3.2), (3.5) and (3.6) we obtain (i), (ii), (iii) and (iv) respectively.

Let □ be a 12-weighting line having on it □ points of weight 2, □ points of weight 1 and □ points of weight 0. Then, counting points on □ gives

$$□ + □ + □ = □ + 1 = 12 \quad (4.1)$$

and summing the weights of points on □ gives

$$2□ + □ = □ - 11 = 12 \quad (4.2)$$

So the possible non-negative integers solutions of the above equations are 7.

Lemma 4.3. (i) The points of weight two form a $(66, 11)$ −arc of type (12,0,0,0,0,55,66,0,0,0,0,0).

(ii) The points of weight one form a $(12, 12)$ −arc of type (0,0,0,0,0,0,0,0,0,0,66,12,55).

(iii)The points of weight zero form a $(55, 6)$ −arc of type (55,66,0,0,0,0,12).

Proof. Directly, from the equations (3.3) and (3.4).

For (ii) and (iii), since $□ \leq 6$, $□ \leq 12$, and $□ \leq 6$, then we get the requirements. □

Remark. (i) From Lemma 2.7, the number of points which pass through their two □-weighting lines is 66 and equal to the number of points of weight 2.

(ii) The number of points which pass through their one □-weighting line is 12 and equal to the number of points of weight 1.

From Lemma 2. 12, we have $□ = \sum_{□=1}^{2} □_{□} = 78$.

Hence we deduce the following theorem.

Theorem 4.4. There exist a $(78, 23;\ □)$ −arc of type $(12, 23)$ in □□$(2, 11)$ when the points of weight 0 is 55 and the points of weight 1 form an envelope. □

For the case $□_0 = 56, 58, 61$ and 65

From equation 3.7, by putting $□_0 = 56, 58, 61$ and 65 we have $□ = 22, 21, 20$ and 19 respectively.

From Lemma 2.12, $□ = 22, 21, 20$ and 19 we have the minimal case W=132, 120, 108 and 96 respectively.

From Lemma 3.3, we have $□_{□}^{2} = 2$, this mean that the number of □-weighting lines form a dual $□_{□}$ −arc and the remaining points of □□$(2,11)$ are classified to the following types:

(i) The points which pass through their two □-weighting line and are represent points of weight two which equal to $\frac{□_{□}(□_{□}-1)}{11}$.

(ii) The points which pass through their one □-weighting line and are represent points of weight one which equal to $□_{□}(□ + 2 - □_{□})$.

Theorem 4.5. There exist a (□, □; □)-arc, of type $(□ - 11, □)$ in □□$(2,11)$ when the points of weight one lies on only one □ −weighting line of (□, □; □)-arc and the points of weight two lies on exactly two □ −weighting linesand the points of weight zero equal $56, 58, 61$ and 65 .

Proof:

Let the number of points of weight zero equal *56*, from Lemma 2. 7, the number of points of weight one equal *22* and the points of weight two equal *55*. And the points of weight two form a $(55, 10)$ −arc, the points of weight one form a $(22, 11)$ −arc and the points of weight zero form a $(56, 6)$ −arc.

Hence $\square = \sum_{\square=1}^{2} \square_{\square} = 77$.

By the same arguments, for the number of points of weight *0* equal 58,61 and 65, the number of points of weight *1* equal 30,36 and 28 and the number of points of weight *2* equal 45,36 and 40, then we get $\square = 75,72$ and *68*respectively. □

References

M. Y. Abass, "Existence and non – existence of (k, n; *f*) – arcs of type $(\square - 3, \square)$ and monoidal arcs in PG(2, 9)", M. Sc. Thesis, University of Basrah, Iraq (2011).

Z. A. Abudljebar,"A generalization of $(k, n, f) -$ arcs of type $(1, n)$ in $PG(2, q)$ and weighted (K, n) −arcs of type $(n - 4, n)$ in PG(2,8)", M.Sc. Thesis, University of Basrah, Iraq (2012).

M. Barnabei, " On arcs with weighted points " , Journal of statistical planning and Inference, 3 (1979), 279 – 286.

E. D'Agostini, " Alcune osservazioni sui (k, n; *f*) – archi di un piano finite ", Atti Accad. Sec.1st. Bologna Rend, 6:211-218,(1979).

E. D'Agostini, "Sulla caratterizzazione delle (k, n; *f*)-calotte di tipo (n –2, n)", Atti Sem. Mat. Fis. Univ. Modena, 29: 263 - 275,(1980).

F. K. Hameed, "Weighted (k, n) – arcs in the projective plane of order nine " , Ph.D. Thesis, University of London, England (1989).

J. W. P. Hirschfeld, "projective geometries over finite fields", Second Edition, Clarendon Press Oxford, 1998.

R. D. Mahmood, " (k, n; *f*) –arcs of type (n – 5, n) in PG(2, 5) ", M.Sc. Thesis, University of Mosul, Iraq (1990).

M. TalliniScafati, "Graphic curves on a Galois plane", In Atti del Convegno di Geometrica Combinatoriae sue Applicazion", pages 413-419. Universita di perugia, 1971.

B. J. Wilson, " (k, n; *f*) – arcs and caps in finite projective spaces", Annals of Discrete mathematics 30 (1986), 355 – 362.

الأقواس الموزونة-(k, n) من النوع (n-11,n) في المستوي الأسقاطي من الرتبة 11

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الخلاصة

في هذا البحث برهنا وجود الأقواس الموزونة من النوع(n-11,n) في المستوي الأسقاطي من الرتبة 11. وجدنا منهجيه جديدة لبناء الأقواس الموزونة من النوعين. لقد ثبتا وجود القوس (78,n;□) من النوع (n-11,n) عندما عدد النقاط التي لها الوزن صفر تساوي (55)،والنقاط التي لها وزن واحد هي envelope(12). و أخيرا، ثبتنا وجود الأقواس الموزونة من النوع (n-11,n) عندما عدد النقاط من الوزن واحد تقع على مستقيم واحد فقط من الوزن n بالأقواس الموزونة و النقاط من الوزن أثنين تقع على مستقيمين من الوزن n .