On Weighted Arcs of Type $(n-q, n)$ in PG(2, q)

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Abstract

In this paper we proved that the existence of minimal ($4q-2, q+4; f$) – arc of type (4, $q+4$), which having W is minimal and there is no point of weight greater than 2 in the projective plane of order q. Also we proved that the existence of $\left(q+1+\frac{q(q+1)}{2}, 2q+1; f\right)$ – arc of type $(q+1, 2q+1)$ in $PG(2,q)$ where the points of weight 1 formed a conic, the points of weight 2 are the exterior points of the conic, the points of weight 0 are the interior points of the conic, W is minimal and $\mathrm{Im}f = \{0,1,2\}$.

Key words. Projective Plane of order q**,** weighted (k,n) –arcs.

Mathematics Subject Classification. 51E21.

Introduction

[A. Barlotti, 1978] presented the notion of a $(k,n;\{w_i\})$ – set of kind s. The $(k,n;\{w_i\})$ – set of kind 2 in a projective plane, also called $(k,n;\{w_i\})$ – arcs, where studied by [M. Barnabei, 1979]. [E. D'Agostini, 1980], was developed the notion of (k, n; f) – arcs of type $(n-2, n)$ in $PG(2,q)$, where q is even. Also [B. J. Wilson, 1986], generalized the work of [E. D'Agostini, 1980], into (k, n; f) – arcs of type $(n-3, n)$ in PG(2, q), where $3|q$. [F. K. Hameed, 1989], [F. K. Hameed et. al, 2011], [M. Y. Abass, 2011] and [R. D. Mahmood, 1990], continue on the same idea of [B. J. Wilson, 1986]. The notion of (k, n; f) – arcs of type $(1, n)$ introduced by [G. Raguso and L. Rella, 1983].

1. Preliminaries

We will denote by $PG(2,q)$ the projective desarguesian plane of order $q = p^h$, by $\mathcal{P}$ the set of all points of the plane and by $\mathcal{R}$ the set of all lines of the plane. Then $PG(2,q)$ having q^2+q+1 points and q^2+q+1 lines. Each line contains $q+1$ points and through every point there pass $q+1$ lines. Also the projective plane has the following properties:

(i) Any two points have exactly one line joining them and any two different lines meet at just one point.
(ii) There exists four points, no three of them are collinear.

Definition 1.1 [J. W. P. Hirschfeld, 1998]. A (k, n) – arc $\mathcal{K}$ in $PG(2,q)$ is a set of k points no $n+1$ of them are collinear, where $n \geq 2$. We will write simply k – arc for $(k, 2)$ – arc.

Definition 1.2 [J. W. P. Hirschfeld, 1998]. A line ℓ in $PG(2,q)$ is an i – secant of a (k, n) – arc $\mathcal{K}$ if $|\ell \cap \mathcal{K}| = i$. Let τ_i denote the total number of i – secants to $\mathcal{K}$ in $PG(2,q)$, then the type of $\mathcal{K}$ is defined by $(\tau_0, \tau_1, \ldots, \tau_n)$.

Lemma1.1[J. W. P. Hirschfeld, 1998]. For a $(k,n) -$ arc $\mathcal{K}$, the following equations hold:

(i) $\sum_{i=0}^{n} \tau_i = \mathrm{q}^2 + \mathrm{q} + 1$;

(ii) $\sum_{i=1}^{n} i\tau_i = k(q+1)$;

(iii) $\sum_{i=2}^{n} \frac{i(i-1)}{2} \tau_i = \frac{k(k-1)}{2}$.

From [J. W. P. Hirschfeld, 1998], we have 2 – secant of the conic ($(q+1)$ – arc) having 2 points of the conic, $\frac{q-1}{2}$ exterior points (the points of intersection of two 1 – secants of the conic) and $\frac{q+1}{2}$ interior points (the points which there pass through them no 1 – secant of the conic). The 1– secants of the conic contain q exterior points and 1 point of the conic. The 0 – secants of the conic contain $\frac{q+1}{2}$ exterior points and $\frac{q+1}{2}$ interior points.

For any function f from $\mathcal{P}$ to the set of natural numbers $\mathbb{N}$ we will say that $f(P)$ is the weight of the point P. By using f, we may define a function F from $\mathcal{R}$ to $\mathbb{N}$ in the following way:

$$F(r) = \sum_{P \in r} f(P)$$

and we will say that $F(r)$ is the weight of the line r. Moreover, if $F(r) = j$ we will also say that r is a j – weighting line.

Definition 1.3 [E. D'Agostini, 1994]. A $(k,n;f)$– arc K in $PG(2,q)$ is a function $f : \mathcal{P} \longrightarrow \mathbb{N}$ such that $k = |$support of f (the points of non – zero weight) $|$ and $n = \max F$.

Let us remark that an ordinary $(k,n) -$ arc is a $(k,n;f)$– arc with $\operatorname{Im} f = \{0, 1\}$.

Definition 1.4 [F. K. Hameed, 1989]. For any $(k,n;f)$– arc K the underlying arc is the ordinary arc whose points are all the points of K.

Definition 1.5 [F. K. Hameed, 2012]. A $(k,n;f)$– arc K is called monoidal if Im f = { 0, 1, ω } and $l_\omega = 1$.

Definition 1.6 [E. D'Agostini, 1994]. The characters of a $(k, n; f)$– arc K are the integers $t_j = |F^{-1}(j)|$; $j = 0, 1, \dots, n$.

Definition 1.7 [E. D'Agostini, 1994]. The type of a $(k, n; f)$– arc K is the set of $\text{Im}\, F$. To write explicitly the type of K we can use the sequence $(n_1, \dots, n_\rho)$, where $n_\lambda \in Im\, F$, $\lambda = 1, \dots, \rho$ and $n_1 < n_2 < \cdots < n_\rho = n$.

Let us use the following notations:

$\omega = \max f$, $W = \sum_{P \in \mathcal{P}} f(P)$ and W will be called the weight of K.

$L_i = f^{-1}(i)$ and $l_i = |L_i|$, $i = 0, 1, \dots, \omega$.

$[\, M \,]$ indicates the set of all lines through the point M .

It is well known from [E. D'Agostini, 1994], that:

$$k = \sum_{i=1}^{\omega} l_i \tag{1.1}$$

$$W = \sum_{i=1}^{\omega} i\, l_i \tag{1.2}$$

$$\sum_{r \in [M]} F(r) = W + q\, f(M) \tag{1.3}$$

$$|Im\, F| \geq 2 \tag{1.4}$$

A useful result, mentioned in [E. D'Agostini, 1994], is the following:

If there exists a point P of a $(k, n; f)$– arc K such that every line through it is an n – weighting line, then

$$P \in L_\omega \tag{1.5}$$

If $M \in L_\omega$ and $u \in [\, M \,]$ then,

$$F(u) = n. \tag{1.6}$$

$$\text{Hence} \quad W \leq (n - \omega)q + n .$$

An arc with weight such that the equality holds, is called maximal. Of course, a maximal arc is also such that through a point of maximal weight there pass only n – weighting lines.

Finally, we shall recall [E. D'Agostini, 1994] the following relations concerning the characters of $(k, n; f)$– arc K :

$$\sum_{j=0}^{n} t_j = q^2 + q + 1 \tag{1.7}$$

$$\sum_{j=1}^{n} j\, t_j = (q+1)W \tag{1.8}$$

$$\sum_{j=2}^{n} \binom{j}{2} t_j = \binom{W}{2} + q \sum_{i=2}^{\omega} \binom{i}{2} l_i \tag{1.9}$$

2. (k, n; *f*) – arcs of type (m, n)

From now on, K shall denote a $(k, n; f)$– arc of type $(m,\ n)$, where $|Im\ f| \geq 3$. Let firstly state the following:

Lemma 2.1 [F. K. Hameed, 1989]. The weight W of a $(k,n;f)-arc$ of type (m,n) satisfies:

$$m(q+1) \leq W \leq (n-\omega)q + n .$$

We call arcs for which the values in Lemma 2.1 are attained, maximal and minimal $(k,n;f)-$ arcs of type (m,n) respectively.

Theorem 2.1 [F. K. Hameed, 1989]. Let K be a $(k,n;f)-arc$ of type (m,n) , $m > 0$ and let v_m^s and v_n^s respectively the number of lines of weight m and the number of lines of weight n passing through a point of weight s . Then

$$(n-m)v_m^s = (n-s)(q+1) - (W-s) ;$$
$$(n-m)v_n^s = (W-s) - (m-s)(q+1) .$$

Theorem 2.2 [F. K. Hameed, 1989]. The necessary condition for the existence of a $(k,n;f)-arc$ K of type (m,n) , $m > 0$ is that:

(i) $q \equiv 0 \ mod(n-m)$;

(ii) $\omega \leq n-m$;

(iii) $m \leq n-2$.

3. (k, n; *f*) – arcs of type (n – q, n)

From [F. K. Hameed, 1989], we have the following:

(i) $\sum_{r\in[M]} F(r) = W + q\, f(M)$;

(ii) $(n-q)(q+1) \leq W \leq (n-\omega)q + n$.

Lemma 3.1. In the (k, n; f) – arcs of type $(n-q, n)$ in $PG(2,q)$, if $W = (n-q)(q+1)$, then

(i) $v_{n-q}^s = (q-s)+1$;
$v_n^s = s$.

For $s = 0, 1, 2, \dots, q$.

(ii) $t_n = n-q$;
$t_{n-q} = q^2 + 2q + 1 - n$.

Proof. (i) We have $v_{n-q}^s + v_n^s = q+1$ and if $f(M) = s$, then

$$(n-q)v_{n-q}^s + n v_n^s = \sum_{r\in[M]} F(r) = W + q\, s .$$

Solve these two equations when $W = (n-q)(q+1)$, give the result.

(ii) From the following equations

$$\sum_{j=0}^{n} t_j = q^2 + q + 1;$$

$$\sum_{j=1}^{n} j\, t_j = (q+1)W$$

For $j = n - q,\ n$ we obtain the result. □

Now, we take the case in which $l_0 > 0,\ l_1 > 0,\ l_2 > 0$ and $l_s = 0$ for $s = 3, 4, 5, \dots, q$. Let l be an n – weighting line on which by (i) of Lemma (3.1) there are no points of weight 0 and suppose that on l there are α points of weight 1 and β points of weight 2 . Then counting points of l, gives the following:

$$\alpha + \beta = q + 1$$

And counting the weights of points on l, we get

$$\alpha + 2\beta = n$$

Solving these two equations gives

$$\alpha = 2(q + 1) - n \qquad (3.1)$$

And $$\beta = n - (q + 1) \qquad (3.2)$$

Counting incidences between points of weight 2 and n – weighting line gives

$$l_2\, v_n^2 = t_n\, \beta\,.$$

Using Lemma 3.1 and equation (3.2) we get

$$l_2 = \frac{(n-q)(n-q-1)}{2}. \qquad (3.3)$$

And counting incidences between points of weight 1 and n – weighting line gives

$$l_1\, v_n^1 = t_n\, \alpha\,.$$

Hence, by using Lemma 3.1 and equation (3.1) we get

$$l_1 = (n - q)(2q + 2 - n)\,. \qquad (3.4)$$

Making use of the equations (3.3) and (3.4) and the following relation:

$$l_0 + l_1 + l_2 = q^2 + q + 1\,.$$

It follows that,

$$2q^2 + 2q + 2 - (n - q)(3q + 3 - n) - 2l_0 = 0;$$

$$2q^2 + 2q + 2 + (n - q)(n - 3q - 3) - 2l_0 = 0;$$

$$n^2 - (4q + 3)n + 5q^2 + 5q + 2 - 2l_0 = 0. \qquad (3.5)$$

If $l_0 = q^2 - 3q + 3$, then from (3.5), we have:

$$n^2 - (4q + 3)n + 3q^2 + 11q - 4 = 0.$$

Thus,

$$n = \frac{\left[(4q + 3) \pm \sqrt{4q^2 - 20q + 25}\ \right]}{2} = \frac{[(4q + 3) \pm (2q - 5)\,]}{2}$$

Therefore, $n = q + 4$ or $n = 3q - 1$.

Lemma 3.2. The value of n in the (k, n ; f) – arcs of type $(n-q, n)$ in $PG(2,q)$, with $W=(n-q)(q+1)$ and Im(f) = {0, 1, 2} satisfies the following inequality:

$$q+1<n<2q+2\,.$$

Proof. For $n \leq q+1$ we get $l_2 \leq 0$. But $l_2 > 0$ and l_2 cannot be zero because this implies that Im(f) = {0, 1} and this contradiction. For $n \geq 2q+2$ we get $l_1 \leq 0$. But $l_1 > 0$ and l_1 cannot be zero because this implies that Im(f) = {0, 2} and this contradiction.

Lemma 3.3. In the (k, n ; f) – arcs of type $(n-q, n)$ in $PG(2,q)$, if $W=(n-q)(q+1)$, Im(f) = {0, 1, 2} and $l_0=q^2-3q+3$, then the value of $n=q+4$.

Proof. If $n=3q-1$, then from Lemma (3.2), we must have $3q-1<2q+2$, i.e. $2q+2-(3q-1)>0$. But $2q+2-(3q-1)=3-q<0$, if $q>3$.

Theorem 3.1. When Im(f) = {0, 1, 2} and $l_0=q^2-3q+3$, there exists ($4q-2, q+4; f$) – arc of type (4, $q+4$) in $PG(2,q)$, having:

(i) $W=4(q+1)$ and $l_1=4q-8$;

(ii) $l_2=6$ and its coordinates points form a (6, 3) – arc with four 3– secants in $PG(2,q)$.

Proof. From Lemma 3.3, we have $n=q+4$, and then from (3.3) and (3.4), we get:

$$l_1=4q-8\,;\ l_2=6\,.$$

Also from Lemma 3.1, we obtain:

$$t_{q+4}=4.$$

From the equations (3.1) and (3.2), we have:

$$\alpha=q-2 \quad ; \qquad \beta=3\,.$$

Then the points of weight 2 form (6, 3) – arc $\mathcal{K}$, has four 3 – secants because there are three of them on $(q+4)$ – weighting line and there are no more than two of them on 4– weighting line. That is $\tau_3=t_{q+4}=4$ and $\tau_0+\tau_1+\tau_2=t_4=q^2+q-3$. Also the points of weight 1 are the remaining points on 3 – secants of $\mathcal{K}$.

Theorem 3.2. There exist $\left(q+1+\frac{q(q+1)}{2}, 2q+1; f\right)$ –arc of type $(q+1, 2q+1)$ in $PG(2,q)$, where the points of weight 1 formed a conic, W is minimal and Im$f=\{0,1,2\}$.

Proof. If $n=2q+1$, then equations (3.3) and (3.4), give:

$$l_2=\frac{q(q+1)}{2} \quad ; \quad l_1=q+1\,.$$

Then the point of weight 1 are the points of the conic in $PG(2,q)$, and the points of weight 2 are the exterior points of the conic. Then the points of weight 0 are the interior points of the conic. The tangent lines of the conic have weight equal $2q+1$, the exterior lines and the bisecant of the conic have weight equal $q+1$.

Hence from Lemma (3.1), part (ii), we have $t_{2q+1}=q+1$ and $t_{q+1}=q^2$.

Since $k = l_1 + l_2 = (q+1) + \frac{q(q+1)}{2}$.

Therefore, there exist $\left(q + 1 + \frac{q(q+1)}{2}, 2q + 1; f\right)$ −arc of type $(q + 1, 2q + 1)$ in $PG(2, q)$.

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حول الأقواس الموزونة من النوع $(n-q,n)$ في $PG(2,q)$

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الخلاصة:

في هذا البحث برهنا وجود اصغر قوس – $(4q-2,\ q+4;\ f)$ من النوع $(4,q+4)$ الذي له اصغر W و لا يحتوي على نقطة وزنها اكبر من 2 في المستوي الاسقاطي من الرتبة q . كذلك برهنا وجود القوس – $\left(q+1+\frac{q(q+1)}{2},2q+1;f\right)$ من النوع $(q+1,2q+1)$ في $PG(2,q)$ حيث النقاط التي لها وزن 1 تشكل قطع مخروطي و النقاط التي لها وزن 2 هي النقاط الخارجية لهذا القطع المخروطي و النقاط التي لها وزن 0 هي النقاط الداخلية للقطع المخروطي و W هو الاصغر و $\mathrm{Im}(f)=\{0,1,2\}$.

الكلمات المفتاحية: المستوي الاسقاطي من الرتبة q ، الاقواس الموزونة – (k,n) .

تصنيف الموضوع رياضياً: 51E21 .